10-Transmission Line Models Part 1

Text: 4.0-4.5

ECEGR 451
Power Systems

Topics

- Distributed Line Model
- Terminal Characteristics
- Traveling waves

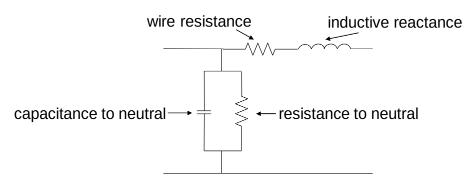
Transmission Line Modeling

- Previously discussed how to determine parameters (inductance, capacitance, resistance) of lines
- We will now form transmission line models
- First consider sinusoidal steady-state
- Let

$\mathbf{z} = r + j\omega l$ series impedance per meter

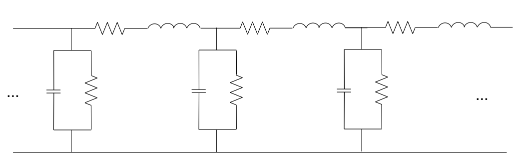
$\mathbf{y} = g + j\omega c$ shunt admittance per meter to neutral

Transmission Line Modeling



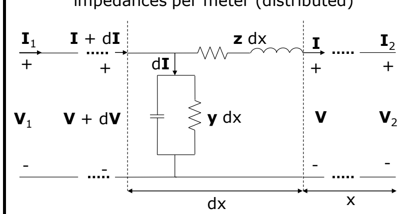
circuit elements

Distributed Model



Transmission Line Terminal Characteristics

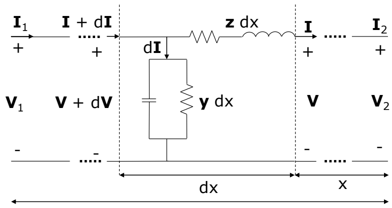
impedances per meter (distributed)



X = length of the line

$\mathbf{V}_1$: sending end
$\mathbf{V}_2$: receiving end

Transmission Line Terminal Characteristics



from KVL & KCL

$d\mathbf{V} = (\mathbf{Iz})dx$

$d\mathbf{I} = (\mathbf{V} + d\mathbf{V})\mathbf{y}\ dx \approx \mathbf{Vy}\ dx$ Ignoring d**V**dx (small value)

Transmission Line Terminal Characteristics

- From previous slide:

$d\mathbf{V} = (\mathbf{Iz})dx$

$d\mathbf{I} = \mathbf{Vy}\ dx$

- Rearranging we get two first-order linear differential equations

$\frac{d\mathbf{V}}{dx} = \mathbf{zI} \qquad \frac{d\mathbf{I}}{dx} = \mathbf{yV}$

or one second-order linear differential equation

$\frac{d^2\mathbf{V}}{dx^2} = \frac{d\mathbf{Iz}}{dx}$

$\frac{d^2\mathbf{V}}{dx^2} = \mathbf{yzV} = \gamma^2\mathbf{V}$

$\gamma \triangleq (\mathbf{yz})^{1/2}$

Transmission Line Terminal Characteristics

- γ is the propagation constant
- Is γ real, imaginary or complex?

$\gamma \triangleq (yz)^{1/2}$

$z = r + j\omega l$

$y = g + j\omega c$

- Complex, but without resistance in z or conductance in y, it is purely imaginary
- We shouldn't expect real solutions to the differential equations

Transmission Line Terminal Characteristics

- Solving homogeneous linear differential equations

$ay'' + by' + cy = 0 \qquad y' = \frac{f(x)}{dx}$

- Try a solution of the form $y = ke^{sx}$, then

$y' = ske^{sx}$

$y'' = s^2ke^{sx}$

- Via substitution:

$aks^2e^{sx} + bkse^{sx} + cke^{sx} = 0$

$ke^{sx}\left(as^2 + bs + c\right) = 0$

$\left(as^2 + bs + c\right) = 0$ characteristic equation, find the roots

Transmission Line Terminal Characteristics

- Now for our problem

$\frac{d^2\mathbf{V}}{dx^2} = \gamma^2\mathbf{V}$

- Write the characteristic equation
  - remember $ay'' + by' + cy = 0$

Transmission Line Terminal Characteristics

- Now for our problem

$\frac{d^2\mathbf{V}}{dx^2} = \gamma^2\mathbf{V}$

- Write the characteristic equation
  - remember $ay'' + by' + cy = 0$
- Solution:
  - rewrite the equation as $\left(as^2 + bs + c\right) = 0$
  - via substitution

$\frac{d^2\mathbf{V}}{dx^2} + 0 - \gamma^2\mathbf{V} = 0$

$s^2 - \gamma^2 = 0$

Transmission Line Terminal Characteristics

- Characteristic equation
  $s^2 - \gamma^2 = 0$
- Roots
  $s_1, s_2 = \pm\gamma$ conjugate complex roots
- General solution

$$\mathbf{V} = k_1 e^{\gamma x} + k_2 e^{-\gamma x}$$
$$= (k_1 + k_2)\frac{e^{\gamma x} + e^{-\gamma x}}{2} + (k_1 - k_2)\frac{e^{\gamma x} - e^{-\gamma x}}{2}$$
$$= \underbrace{K_1 \cosh(\gamma x) + K_2 \sinh(\gamma x)}_{\text{important equation}} \quad \text{where} \quad K_1 = k_1 + k_2, \quad K_2 = k_1 - k_2$$

Determining K_1 Parameter

- To find K_1, K_2 examine boundary conditions
- At receiving end (x = 0)

$\mathbf{V} = \mathbf{V}_2$

$\mathbf{I} = \mathbf{I}_2$

therefore

$$\mathbf{V}_2 = K_1 \cosh(\gamma x) + K_2 \sinh(\gamma x) \quad \begin{cases} \sinh(0) = 0 \\ \cosh(0) = 1 \end{cases}$$

$K_1 = \mathbf{V}_2$

Determining K_2 Parameter

- Now for K_2
- Taking the derivative of the general solution wrt voltage yields:

$$\frac{d\mathbf{V}(0)}{dx} = -\gamma K_1 \sinh(\gamma x) + \gamma K_2 \cosh(\gamma x)$$
$$\frac{d\mathbf{V}(0)}{dx} = \gamma K_2$$

and from KVL (slide 7)

$$\frac{d\mathbf{V}(0)}{dx} = \mathbf{z}\mathbf{I}_2 \quad \text{therefore}$$
$$K_2 = \frac{\mathbf{z}}{\gamma}\mathbf{I}_2$$

Note: $\frac{\mathbf{z}}{\gamma} = \frac{\mathbf{z}}{\sqrt{\mathbf{yz}}} = \frac{\sqrt{\mathbf{z}}}{\sqrt{\mathbf{y}}} = \left(\frac{\mathbf{z}}{\mathbf{y}}\right)^{1/2}$

Characteristic Impedance

- Define the characteristic impedance as:

$$\mathbf{Z}_c \triangleq \left(\mathbf{z}/\mathbf{y}\right)^{1/2}$$

- General solution for voltage:

$$\mathbf{V} = \mathbf{V}_2 \cosh(\gamma x) + \mathbf{Z}_c\mathbf{I}_2 \sinh(\gamma x)$$

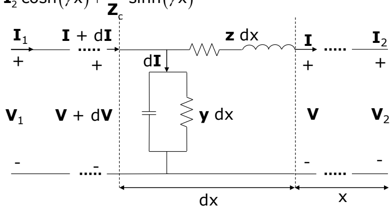
- By similar derivation, general solution for current:

$$\mathbf{I} = \mathbf{I}_2 \cosh(\gamma x) + \frac{\mathbf{V}_2}{\mathbf{Z}_c} \sinh(\gamma x)$$

Transmission Line Terminal Characteristics

Terminal conditions (**I**, **V** when x =X)

$$\mathbf{V} = \mathbf{V}_2 \cosh(\gamma x) + \mathbf{Z}_c\mathbf{I}_2 \sinh(\gamma x)$$
$$\mathbf{I} = \mathbf{I}_2 \cosh(\gamma x) + \frac{\mathbf{V}_2}{\mathbf{Z}_c} \sinh(\gamma x)$$



Example

- Consider
  - 125 MW load at 215 kV and unity PF at 60 Hz
  - transmission line
    - *Rook*
    - 230 miles long
    - 23.8 ft between conductors, horizontally spaced
    - assume line is at 50 degrees C
- Find the voltage, current and power at the sending end

Example

- Start with the receiving end voltage and current
- Now model transmission line

Example

- Start with the receiving end voltage and current

$$\mathbf{V}_2 = \frac{215{,}000}{\sqrt{3}} = 124{,}130\angle 0° \text{ V}$$

$$\mathbf{I}_2 = \frac{125{,}000{,}000}{3(124{,}130)} = 335.7\angle 0° \text{ A} \quad \text{(unity PF)}$$

- Now model transmission line

$$D_m = \sqrt[3]{23.8(23.8)(47.6)} = 30$$

Example

- From Appendix 8
  - resistance = 0.160 Ω/mi @ 50 deg C
  - series reactance = *j*(0.415 + 0.4127) Ω/mi
  - shunt admittance = (10^{-6}) x *j*/(0.0950 + 0.1009) $(\Omega \text{xmi})^{-1}$

$$\mathbf{z} = 0.8431\angle 79.04° \ \Omega/\text{mi}$$

$$\mathbf{y} = 5.105 \text{ x } 10^{-6}\angle 90° \ \text{S/mi}$$

$$\gamma X = \sqrt{\mathbf{yz}}X = 230\sqrt{0.8431(5.105)(10^{-6})}\angle\frac{79.04+90}{2} = 0.4772\angle 84.52°$$

$$\mathbf{Z}_c = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} = 406.4\angle -5.48° = 404.5 - j38.8\Omega/\text{mi}$$

Example

- Find sending end voltage and current using

$$\mathbf{V} = \mathbf{V}_2\cosh(\gamma x) + \mathbf{Z}_c\mathbf{I}_2\sinh(\gamma x)$$

$$\mathbf{I} = \mathbf{I}_2\cosh(\gamma x) + \frac{\mathbf{V}_2}{\mathbf{Z}_c}\sinh(\gamma x)$$

Example

- Find sending end voltage and current using

$$\mathbf{V} = \mathbf{V}_2\cosh(\gamma x) + \mathbf{Z}_c\mathbf{I}_2\sinh(\gamma x)$$

$$\mathbf{I} = \mathbf{I}_2\cosh(\gamma x) + \frac{\mathbf{V}_2}{\mathbf{Z}_c}\sinh(\gamma x)$$

$$\mathbf{Z}_c = 406.4\angle -5.48° \qquad \mathbf{V}_2 = 124{,}130\angle 0 \text{ V}$$

$$\gamma X = 0.4772\angle 84.52° \qquad \mathbf{I}_2 = 335.7\angle 0 \text{ A}$$

$$\mathbf{V}_1 = 137{,}860\angle 27.77° \text{ V}$$

$$\mathbf{I}_1 = 332.31\angle 26.33° \text{ A}$$

Example

Find the power at the sending end

$$\text{PF} = \cos(27.77 - 26.33) \approx 1.0$$

$$P_{3\Phi} = 3\frac{(238{,}780)}{\sqrt{3}}(332.3)(1) = 137{,}443 \text{ kW}$$

Waves on Transmission Lines

- Recall the general solution was of the form
  $\mathbf{V} = k_1 e^{\gamma x} + k_2 e^{-\gamma x}$
- This is consistent with the propagation of wave: voltage travels as a wave down the transmission line

$$\mathbf{V} = k_1 e^{\gamma x} + k_2 e^{-\gamma x}$$

incident wave ↗ ↖ reflected wave

Waves on Transmission Lines

- Voltage can be expressed in terms of location on the line and time
  $\mathbf{V} = k_1 e^{\gamma x} + k_2 e^{-\gamma x}$
- Let
  $\gamma \triangleq \alpha + j\beta$ (recall that γ is complex)
  $\mathbf{V} = k_1 e^{(\alpha + j\beta)x} + k_2 e^{-(\alpha + j\beta)x}$
  $\mathbf{V} = k_1 e^{\alpha x} e^{(j\beta)x} + k_2 e^{-\alpha x} e^{(-j\beta)x}$
- Using the inverse phasor transform:
  $v(t,x) = \sqrt{2}\,\mathrm{Re}\left\{k_1 e^{\alpha x} e^{j(\omega t + \beta x)}\right\} + \sqrt{2}\,\mathrm{Re}\left\{k_2 e^{-\alpha x} e^{j(\omega t - \beta x)}\right\}$

Waves on Transmission Lines

- Voltage travels as wave on the transmission line
  $v(t,x) = \sqrt{2}\,\mathrm{Re}\left\{k_1 e^{\alpha x} e^{j(\omega t + \beta x)}\right\} + \sqrt{2}\,\mathrm{Re}\left\{k_2 e^{-\alpha x} e^{j(\omega t - \beta x)}\right\}$
  $v(t,x) = v_1(t,x) + v_2(t,x)$ (a function of time and position)
- A side note:
  $\alpha \geq 0$ attenuation constant, usually small
  β phase constant

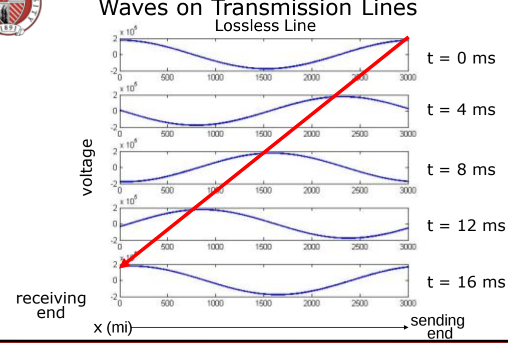
Waves on Transmission Lines

- Observe that if α is ignored, then for a fixed x, v_2 is a sinusoidal function of t
- For a fixed t, v_2 is a sinusoidal function of *x*
  $v(t,x) = \sqrt{2}\,\mathrm{Re}\left\{k_1 e^{\alpha x} e^{j(\omega t + \beta x)}\right\} + \sqrt{2}\,\mathrm{Re}\left\{k_2 e^{-\alpha x} e^{j(\omega t - \beta x)}\right\}$
- Also note that if you scan the line at a constant velocity, v_2 remains constant
  $\omega t - \beta x = \text{constant}$
  $\frac{dx}{dt} = \frac{\omega}{\beta} = \frac{\omega}{\mathrm{Im}\left(\sqrt{zy}\right)}$ velocity of voltage wave

Waves on Transmission Lines

- Consider a transmission line and load with the following parameters:
  - f = 60 hz
  - z = 0.169 + j0.789 Ohms/mi
  - y = 0.538 x 10^{-6} S/mi
  - X = 3,000 mi (a very long line!)
  - $|\mathbf{V}_2|$ = 216,500 V
  - $|\mathbf{I}_2|$ = 326.4 A
- How does the voltage profile look along the line, and w.r.t. time?
  - lossless case
  - lossy case

Waves on Transmission Lines

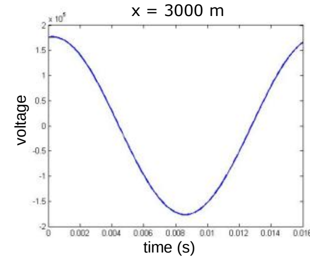


Waves on Transmission Lines

- What is the velocity of the wave?
- From the figure:
  $\frac{dx}{dt} = \frac{3000}{.01667} \approx 180{,}000 \text{ mi/s}$
- From the formula
  $\frac{\omega}{\mathrm{Im}\left(\sqrt{zy}\right)} = \frac{377}{0.0021} \approx 180{,}000 \text{ mi/s}$
- How does this compare with the speed of light?

Waves on Transmission Lines

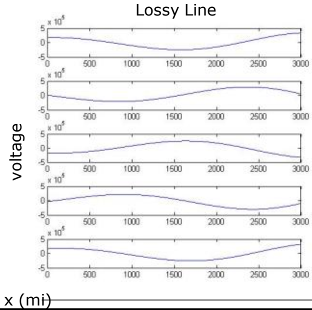
Fixed Position
x = 3000 m



sinusoidal!

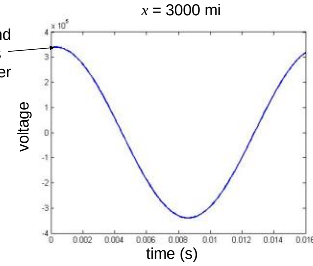
Waves on Transmission Lines

Lossy Line



Waves on Transmission Lines

x = 3000 mi



Summary

- Modeled transmission line using capacitance, inductance and resistance parameters
- Distributed parameters

$$\mathbf{V} = \mathbf{V}_2 \cosh(\gamma x) + \mathbf{Z}_c \mathbf{I}_2 \sinh(\gamma x)$$

$$\mathbf{I} = \mathbf{I}_2 \cosh(\gamma x) + \frac{\mathbf{V}_2}{\mathbf{Z}_c} \sinh(\gamma x)$$

- Voltage along the line behaves like a wave which travels near the speed of light