09-Circuit Theorems

Text: 4.1 – 4.3, 4.8

ECEGR 210

Electric Circuits I

Overview

- Introduction
- Linearity
- Superposition
- Maximum Power Transfer

Introduction

- Nodal and mesh analysis can be tedious to apply to large circuits, especially with multiple sources
- Linearity of the circuits considered can be exploited to simplify analysis

Linearity

- Circuits considered in the this class are linear
- An equation f(x) or system is linear iff
  - Homogeneity: $f(ax) = af(x)$
  - Additivity: $f(x + y) = f(x) + f(y)$

Linearity

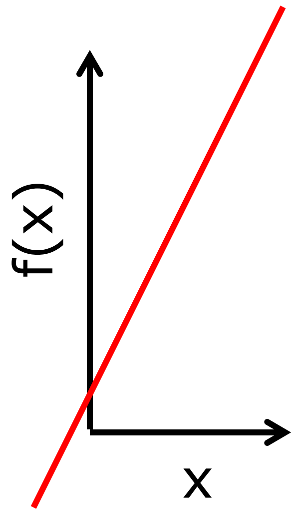
- Is f(x) = 2x + 1 linear?
- Check homogeneity: f(ax) = af(x)

$$2ax + 1 \neq a(2x + 1)$$

- Check additivity : f(x + y) = f(x) + f(y)

$$2(x + y) + 1 \neq 2x + 1 + 2y + 1$$

- No. This is an affine function



Linearity

- What about Ohm's Law (V=IR)? Let I be the argument so that f(x) = V(I)
- Check homogeneity: f(ax) = af(x)

$$aIR = a(IR)$$

- Check additivity: f(x + y) = f(x) + f(y)

$$(I_a + I_b)R = I_aR + I_bR$$

- Yes. A circuit is linear if it consists only of linear elements, linear dependent sources and independent sources

Linearity

- What about P=IV?
  - Careful. V = IR, so $P = I^2R$
- Check homogeneity: f(ax) = af(x)

$$(aI)^2R \neq aI^2R$$

- Check additivity: f(x + y) = f(x) + f(y)

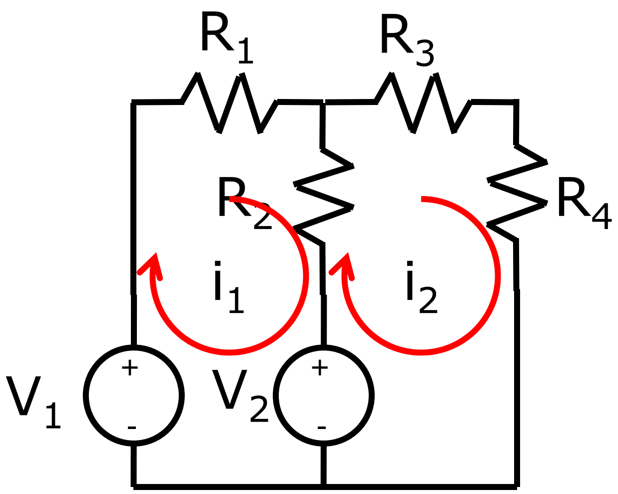
$$(I_1 + I_2)^2R \neq I_1^2R + I_2^2R$$

- No. The power dissipated by a resistor is not linear.

Significance of Linearity

- The mesh currents for the shown circuit can be written as (by inspection):

$$\underbrace{\begin{bmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 + R_4 \end{bmatrix}}_{\mathbf{A}\text{ (matrix)}} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 - V_2 \\ V_2 \end{bmatrix}$$



$$\mathbf{A} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 - V_2 \\ V_2 \end{bmatrix}$$

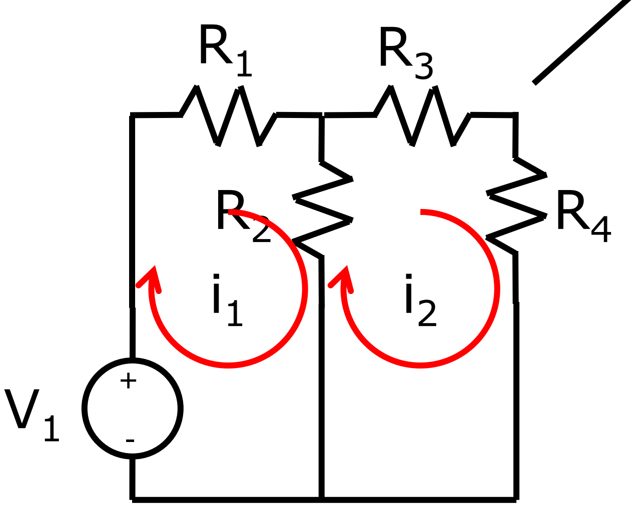
$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} V_1 - V_2 \\ V_2 \end{bmatrix}$$

Solving for i_1, i_2

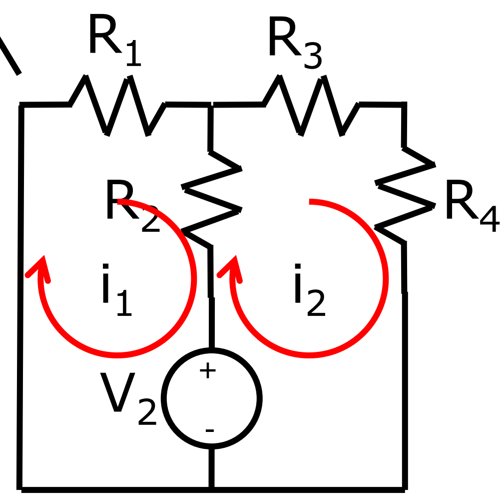
Significance of Linearity

- Observing that:

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} V_1 - V_2 \\ V_2 \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} V_1 - 0 \\ 0 \end{bmatrix} + \mathbf{A}^{-1} \begin{bmatrix} 0 - V_2 \\ V_2 \end{bmatrix}$$

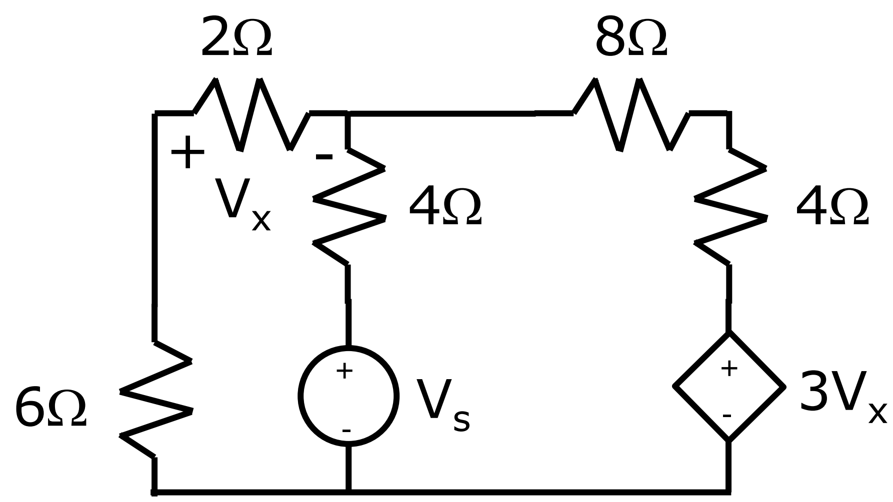


We can solve two "easier" circuits, and add the results!



Significance of Linearity

- Find the current out of the dependent voltage source if V_s = 12V, and if V_s = 24V
  - Should we use nodal or mesh analysis?



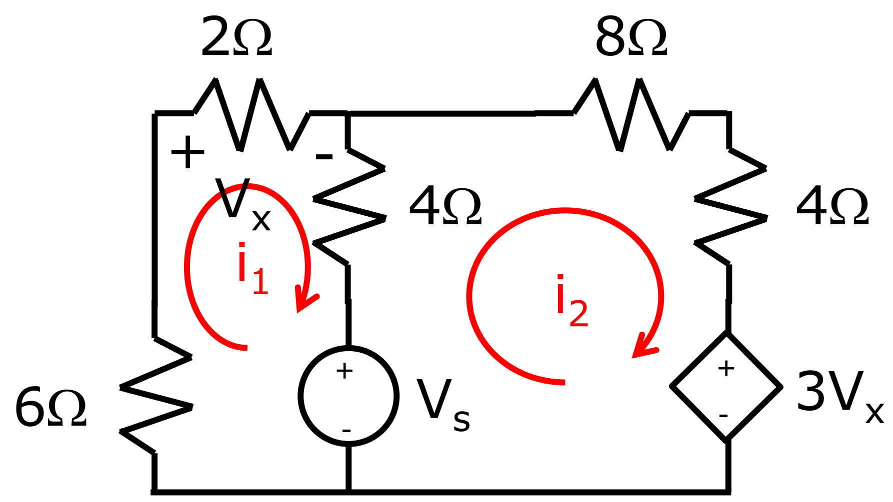
Significance of Linearity

- since $V_s = 12V$

$$0 = 12i_1 - 4i_2 + V_s \text{ (mesh 1)}$$

$$0 = -4i_1 + 16i_2 - 3V_x \text{ (mesh 2)}$$

$$v_x = 2i_1$$



Significance of Linearity

$V_s = 12V$

$0 = 12i_1 - 4i_2 + V_s$

$0 = -10i_1 + 16i_2$

adding...

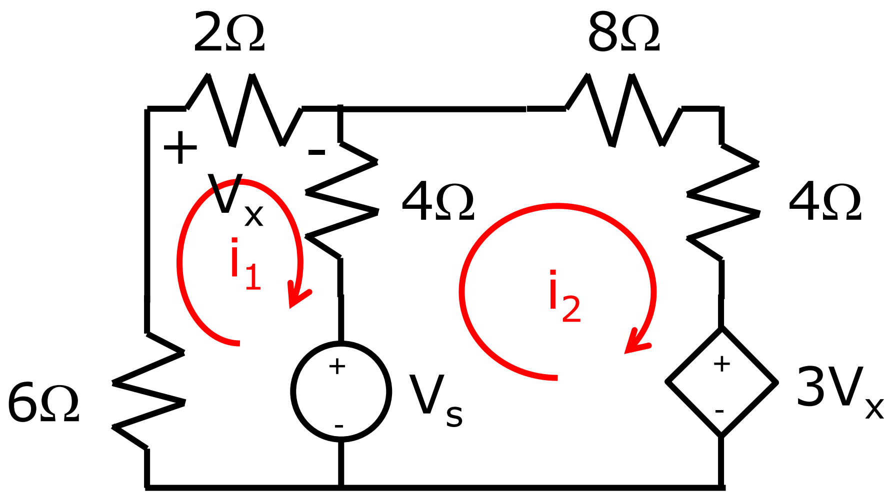
$2i_1 + 12i_2 = 0$

$i_1 = -6i_2$

via substitution

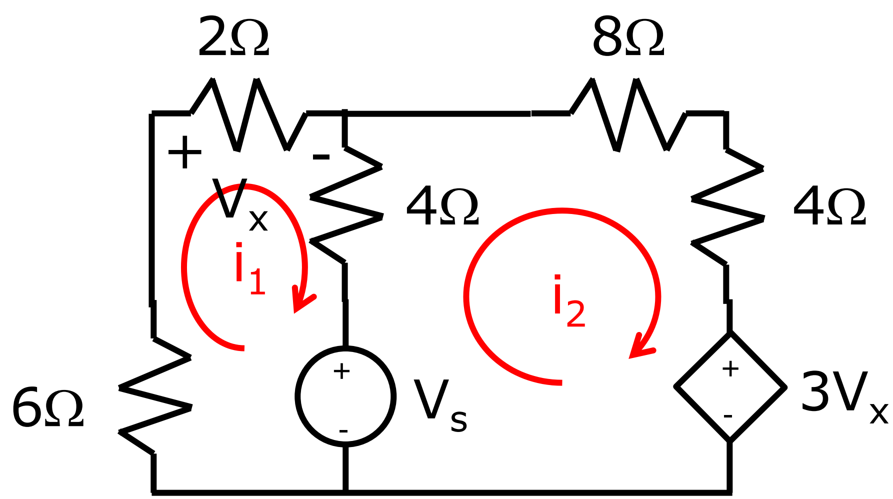
$-76i_2 + V_s = 0$

$i_2 = \frac{V_s}{76} = \frac{12}{76} = 0.158A$



Significance of Linearity

- By linearity: if V_s = 24V (voltage doubled)
  - i_2 = 2x0.158A = 0.316A (current doubled)
- No need to re-solve the circuit
- Behold, the power of linearity!



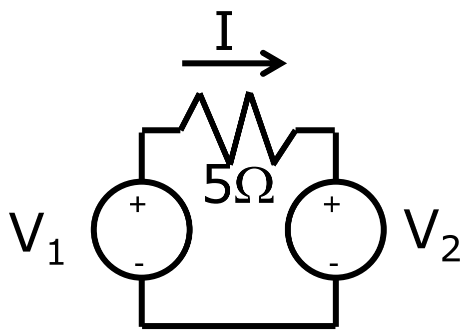
Superposition

- One of the most important circuit analysis benefits of linearity is that superposition can be applied
- Very useful for circuits with multiple independent sources
- Basic idea: analyze the circuit considering only one source at a time, and sum the results once all sources have been considered
  - Valid for all circuit analysis techniques: Nodal, mesh, Ohm's Law, etc.

Superposition

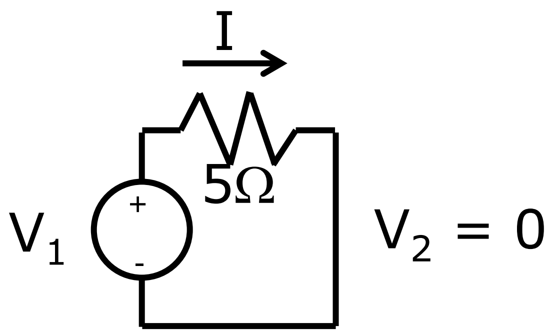
- Simple example with $V_1 = 10V$, $V_2 = 15V$
- Find I

$$I = \frac{V_1 - V_2}{R} = -1A$$



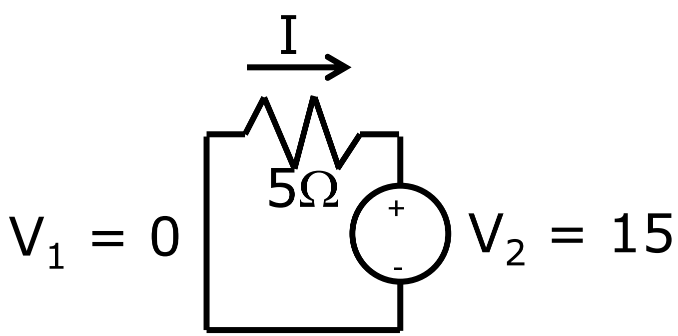
Superposition

- Now solve using superposition
- Consider V_1 only
  - Remove contribution from V_2 (set source to 0, a short circuit)
  - I_{V1} = 2A



Superposition

- Consider V_2 only
  - Remove contribution from V_1 (set source to 0, a short circuit)
  - I_{V2} = -3A



Superposition

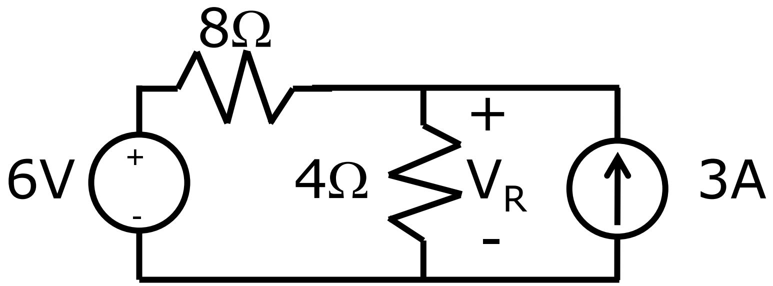
- Combining
  - $I = I_{V2} + I_{V2} = -1A$
  - Same result as solving the circuit simultaneously

Superposition

- Analyze circuit with only one independent source at a time
- Independent voltage sources are ignored by replacing them with a short (voltage source where $V_s = 0$)
- Independent current sources are ignored by replacing them with an open circuit (current source where $I_s = 0$)
- <u>Dependent sources are not ignored</u>
- Sum individual results to analyze circuit

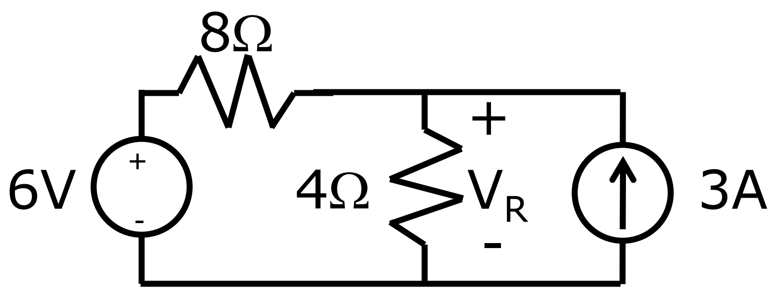
Example

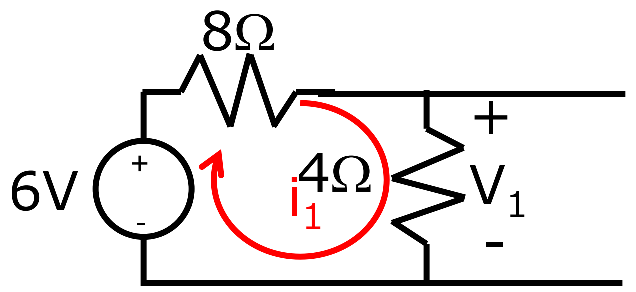
- Find V_R using superposition.



Example

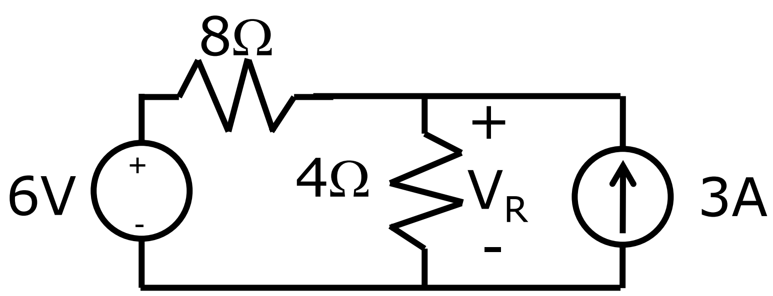
- Consider the voltage source first. Set current source to 0A (open circuit)
- By KVL:
  - $12i_1 - 6 = 0$
  - $i_1 = 0.5A$
  - $V_1 = 2V$ (V_R due to voltage source)

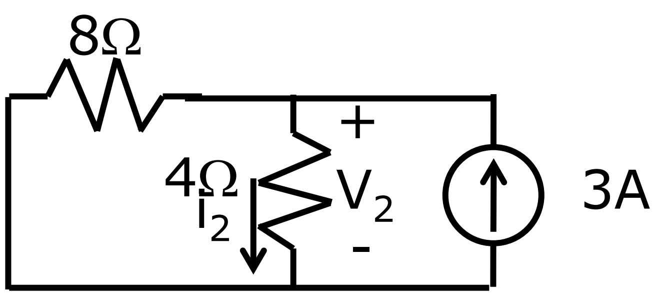




Example

- Now consider the current source. Set voltage source to 0V (short circuit)
- By current division:
  - i_2 = (3x8)/(12) = 2A
  - V_2 = 8V (V_R due to current source)





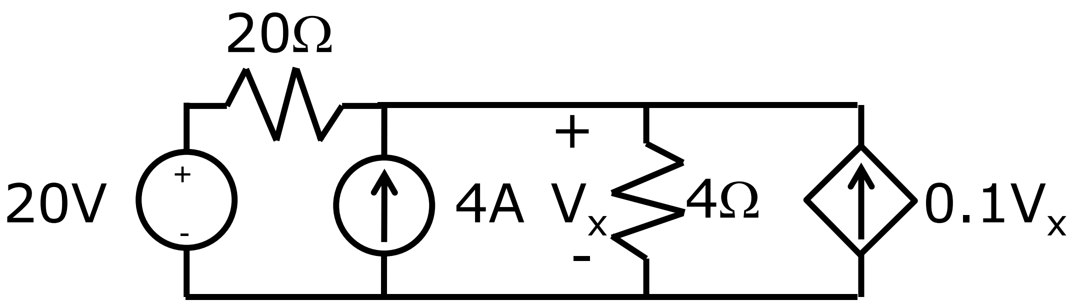
Example

- Now sum results:
  - $V_R = V_1 + V_2 = 10V$
- Note:
  - $i_R = i_1 + i_2 = 2.5A$
  - $P_R = i_R{}^2R = (i_1 + i_2)^2R = 31.25W$

    $P_R \neq P_1^2 + P_2^2$
  - Any <u>power calculations must be done after all sources considered</u>

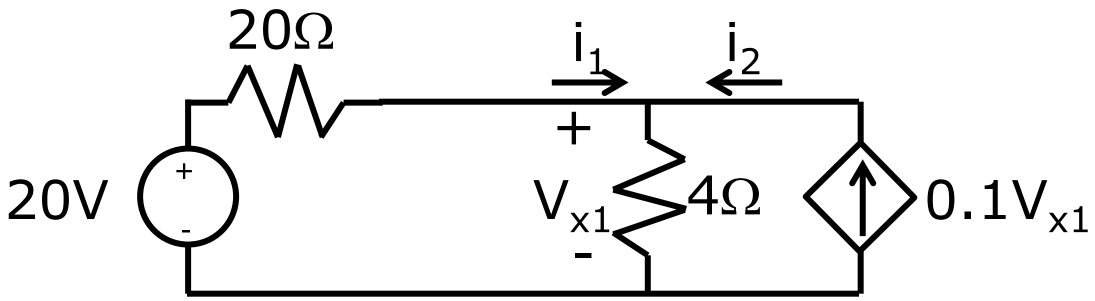
Example

- Use superposition to find V_x



Example

- First consider voltage source.
- Use nodal analysis
  - Trying to find node voltage
  - Only one node with unknown voltage
  - Many elements in parallel



Example

- KCL of node

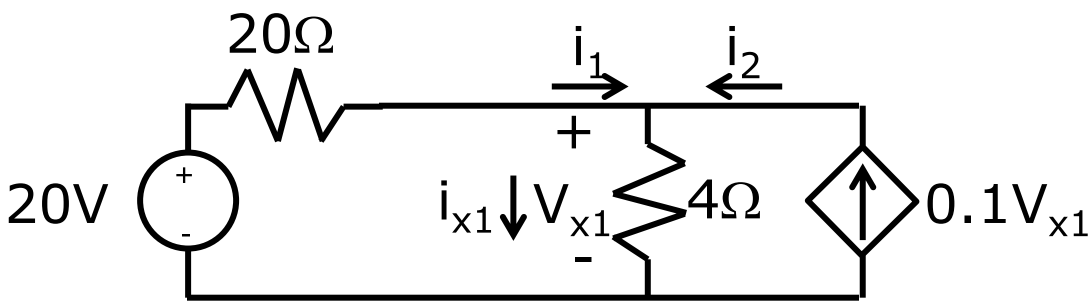
$$i_{x1} = i_1 + i_2$$

via substitution:

$$\frac{V_{x1}}{4} = \frac{20 - V_{x1}}{20} + 0.1V_{x1}$$

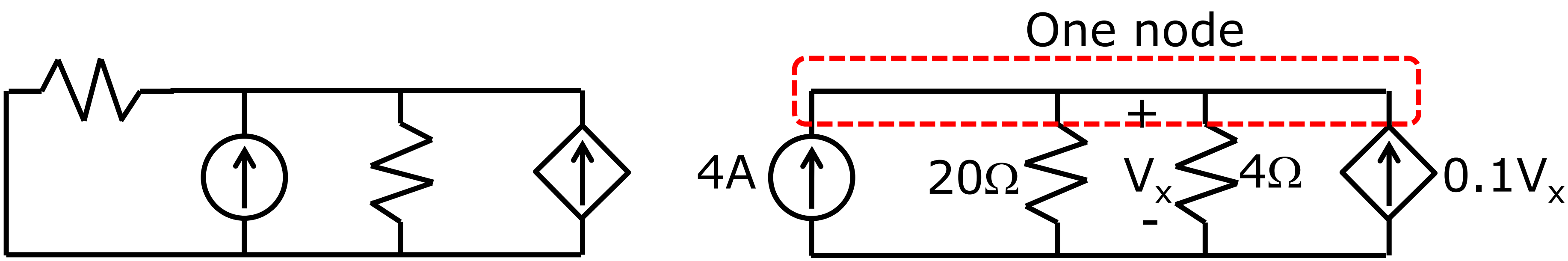
$$V_{x1} - 0.4V_{x1} + 0.2V_{x1} = 4$$

$$V_{x1} = 5V$$



Example

- Now consider current source.
- Rearrange parallel elements for clarity
- Use nodal analysis
  - Trying to find node voltage
  - Only one node with unknown voltage
  - Many elements in parallel



Example

- Now consider current source:

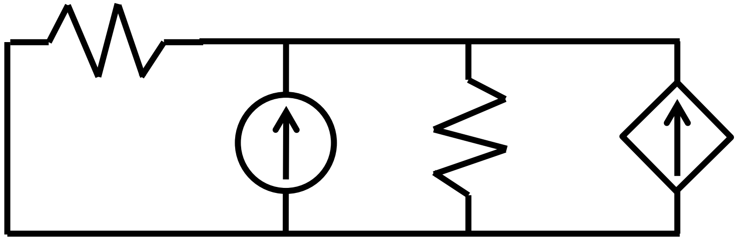
$$4 + 0.1V_{x2} = i_{20} + i_{x2}$$

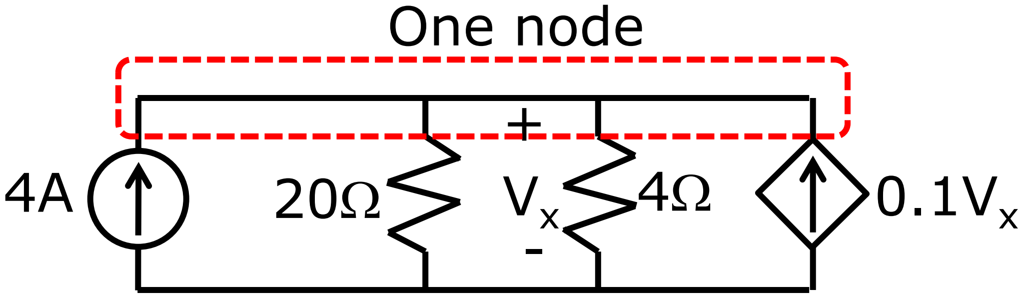
via substitution:

$$4 + 0.1V_{x2} = \frac{V_{x2}}{20} + \frac{V_{x2}}{4}$$

$$4 = 0.05V_{x2} + 0.25V_{x2} - 0.1V_{x2}$$

$$V_{x2} = 20V$$





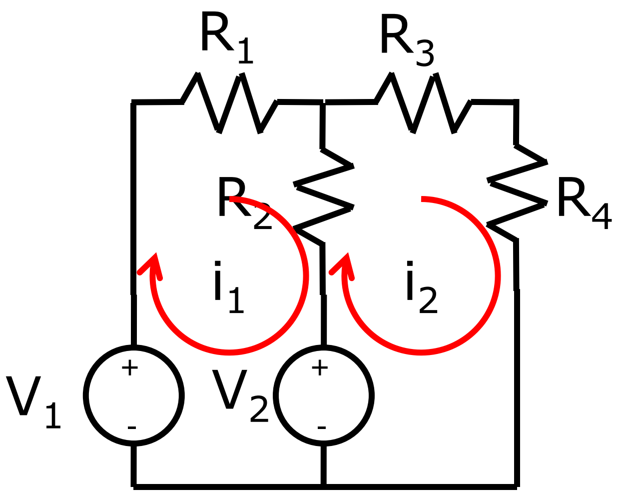
Example

- Summing results:
  - $V_x = V_{x1} + V_{x2} = 25V$

Significance of Linearity

- Consider past example

$$\underbrace{\begin{bmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 + R_4 \end{bmatrix}}_{\mathbf{A}\ \text{(matrix)}} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 - V_2 \\ V_2 \end{bmatrix}$$



Significance of Linearity

- Assume that we solved for i_1 and i_2
- If the voltage sources are increased by a factor of x, what happens to i_1 and i_2?

$$\mathbf{A}\begin{bmatrix} i_1^{new} \\ i_2^{new} \end{bmatrix} = x\begin{bmatrix} V_1 - V_2 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} i_1^{new} \\ i_2^{new} \end{bmatrix} = x\underbrace{\mathbf{A}^{-1}\begin{bmatrix} V_1 - V_2 \\ V_2 \end{bmatrix}}_{\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}}$$

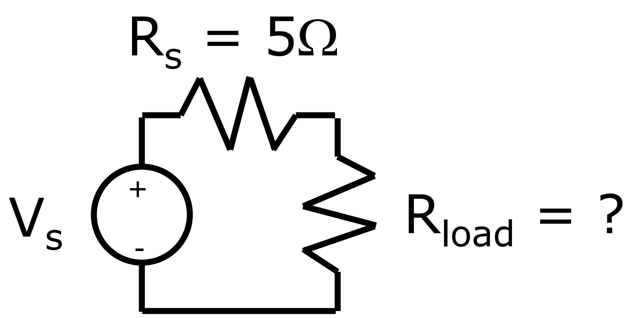
$$\begin{bmatrix} i_1^{new} \\ i_2^{new} \end{bmatrix} = x\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$ Current also increases by x

Significance of Linearity

- If all the voltages are proportionally increased/decreased, then all current will also proportionally increase/decrease
  - What happens if all the resistances are proportionally changed?
- Useful if you have already solved the circuit and want to examine how changes will affect the solution

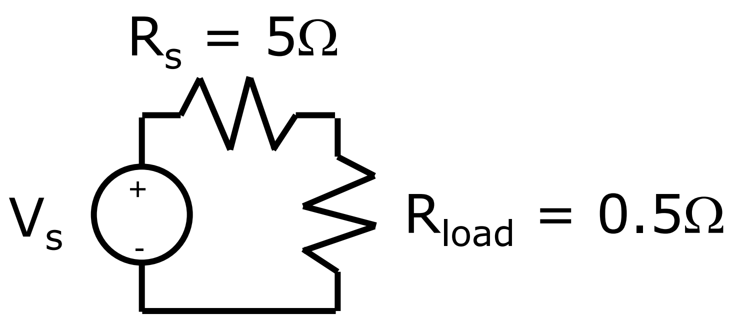
Maximum Power Transfer

- Often interested in designing a circuit so that the maximum power is delivered (transferred) to a load
- Consider an 11V battery connected to a heater by way of a cable
- What resistance should R_{load} be to maximize the power it consumes?



Maximum Power Transfer

- Idea: try a small resistance in order to increase current ($P = I^2R$)
  - Let $R_{load} = 0.5\Omega$
- Compute power to the load resistor and power consumed by the cable

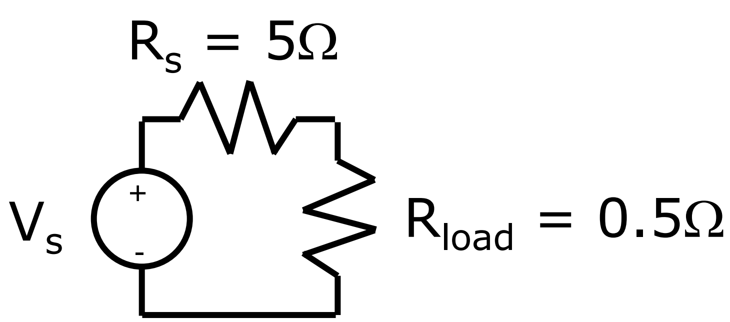


Maximum Power Transfer

$$P_{load} = I^2 R_{load} = \left(\frac{V_s}{R_{load} + R_s}\right)^2 R_{load} = \left(\frac{11}{0.5+5}\right)^2 0.5 = \boxed{2W}$$

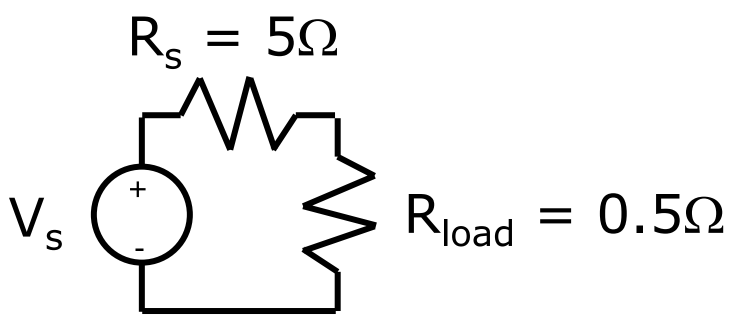
$$P_s = I^2 R_{load} = \left(\frac{11}{0.5+5}\right)^2 5 = \boxed{10W}$$

- More power is consumed by the cable than the load
- Only small portion of applied voltage is across the load
  - $V_{load} = 1V$



Maximum Power Transfer

- New idea: try large resistance so a large voltage appears across the load ($P = V^2/R$)
  - Let $R_{load} = 50\Omega$
- Compute power to the load resistor and power consumed by the cable

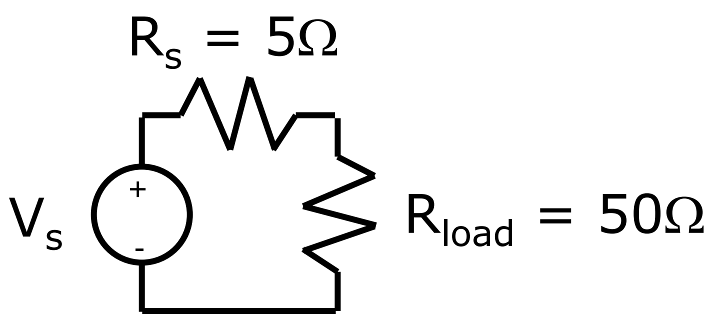


Maximum Power Transfer

$$P_{load} = I^2 R_{load} = \left(\frac{11}{5+50}\right)^2 50 = 2W$$

$$P_s = I^2 R_{load} = \left(\frac{11}{5+50}\right)^2 5 = 10W$$

- More power is consumed by the cable than the load
- Small amount of current flows
  - I = 0.2A



Maximum Power Transfer

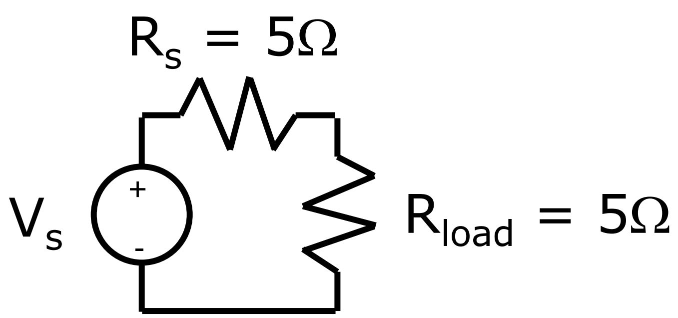
- Trade-off between voltage across load and current through load resistor
- Try $R_{load} = R_s$

Maximum Power Transfer

$$P_{load} = I^2 R_{load} = \left(\frac{11}{5+5}\right)^2 5 = 6.05W$$

$$P_s = I^2 R_{load} = \left(\frac{11}{5+5}\right)^2 5 = 6.05W$$

- Power to load is increased
  - V_{load} = 5.5V
  - I = 1.1A



Maximum Power Transfer

- Maximum power transfer occurs when $R_{load} = R_s$
- Load and source (with series impedance) are said to be "matched"
- Observations
  - Voltage across load is one half applied power
  - Relationship between R_{load} and P_{load} must be non-linear

Maximum Power Transfer

- Proof:

$$P_{load} = I^2 R_{load} = \left(\frac{V_s}{R_{load} + R_s}\right)^2 R_{load}$$

at maximum power transfer $\frac{dP_{load}}{dR_{load}} = 0$

solving

$$\frac{dP_{load}}{dR_{load}} = \frac{V_s^2}{(R_{load} + R_s)^2} + \frac{-2V_s^2 R_{load}}{(R_{load} + R_s)^3}$$

$$\frac{dP_{load}}{dR_{load}} = V_s^2 \frac{R_{load} + R_s - 2R_{load}}{(R_{load} + R_s)^3} = 0$$

therefore

$$R_{load} = R_s$$

Maximum Power Transfer