07-Transmission Line Inductance

Text: 3.0 – 3.3

ECEGR 451
Power Systems

Topics

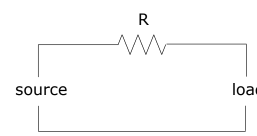
- Inductance
- Single-phase
- N-phase
- Symmetrical spacing

Introduction

- Transmission lines form conductive loops
- Need to model inductance of transmission lines
  - Remember Maxwell's Equations?
- Tedious derivations follow

Transmission Line

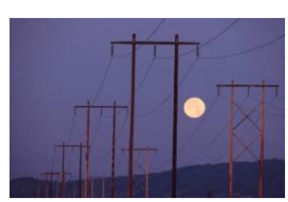
- Our transmission line model so far is limited
- Inductance and capacitance need to be modeled
- Start with inductance

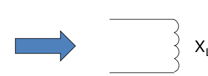


Inductance

- Derive expressions for inductance
  - internal
  - external
  - Geometry-specific
- Incorporate inductance into our line model

Inductance

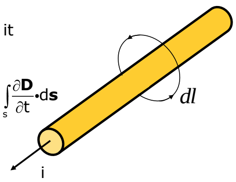




Ampere's Law

- Current through a conductor will produce a circular magnetic field around it
  - Line integral of the magnetic field around the conductor equals the current through it
- Mathematically

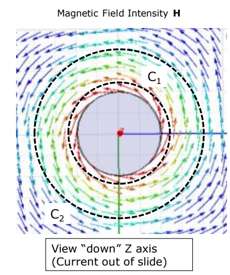
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{or} \quad \oint_C \mathbf{H} \cdot d\ell = \oint_s \mathbf{J} \cdot d\mathbf{s} + \int_s \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$$



Ampere's Law

- Direction of **H** field is CCW
  - Right Hand Rule
- Field strength decreases as path length increases
  - Field becomes weaker as distance increases

$$\oint_{C_1} \mathbf{H} \cdot d\ell = \oint_{C_2} \mathbf{H} \cdot d\ell = \text{constant}$$



Magnetic Flux Density

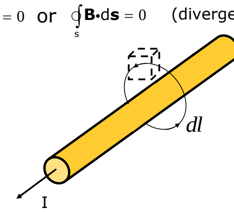
- Related to **H** is the Magnetic Flux Density (**B**)
- In <u>linear</u> isotropic homogeneous (LIH)medium

  $\mathbf{B} = \mu_0 \mathbf{H}$
- where:
  - **B**: magnetic flux density (Wb/m^2)
  - μ_0: permeability of free space 4πE-7 (H/m)
  - μ_r: relative permeability

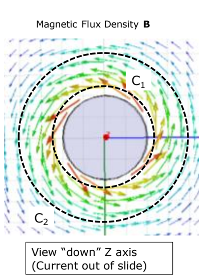
Gauss's Law for Magnetic Fields

- The magnetic flux entering a volume is equal to the magnetic flux leaving it (it is continuous)
- Mathematically

$$\nabla \cdot \mathbf{B} = 0 \quad \text{or} \quad \oint_s \mathbf{B} \cdot d\mathbf{s} = 0 \quad \text{(divergence is 0)}$$



Magnetic Flux Density



- Direction of **B** field is inline with **H** field
- Field strength decreases as path length increases
  - Field becomes weaker as distance increases

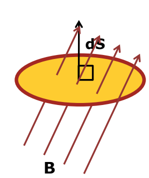
Magnetic Flux

- The total magnetic flux, Φ, passing through a surface is:

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{s}$$

- where:

  Φ: magnetic flux (Wb)



Flux Linkages

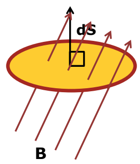
The total flux linking a coil of wire with N turns is

$$\lambda = N\Phi$$

where:

λ: flux linkage (Wb-Turns)

N: number of turns (Turns)



Inductance

Inductance is related to flux linkages as:

$$L \triangleq N\frac{d\phi}{di}$$

$$\lambda = N\Phi$$

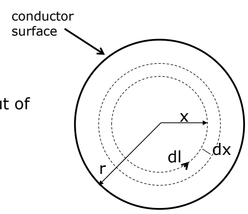
$$\frac{d\lambda}{di} = \frac{Nd\Phi}{di} \quad \text{(derivative wrt i)}$$

$$L = \frac{d\lambda}{di} \quad \text{(via substitution)}$$

We will derive an equation first for internal inductance.
Note: subsequent derivation results in H/m, not H

Internal Inductance

- Assume:
  - uniform current density
  - infinitely long
  - carries total current i (out of slide)
  - i_x current inside radius x



$$i_x = 2\pi x\ H_x \quad \text{from eqn. 1}$$

$$i_x = \frac{\pi x^2}{\pi r^2} i \quad \text{from figure}$$

$$H_x = \frac{x}{2\pi r^2} i \quad \text{solving for } H_x$$

$$B_x = \mu\frac{x}{2\pi r^2} i \quad \text{from eqn. 2}$$

Key Equations

1. $\oint_c H \times dl = i$ 3. $\phi = \int_A B \cdot da$
2. $B = \mu H$

Internal Inductance

$$B_x = \frac{\mu x i}{2\pi r^2} \quad \text{flux density x at meters}$$

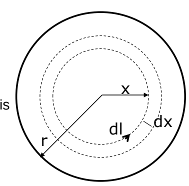
$$d\phi = \frac{\mu x i}{2\pi r^2} dx \quad \text{from eqn 3}$$

$$d\lambda = \frac{\pi x^2}{\pi r^2} d\phi$$ only a fraction of the current is linked (not all of the current is affected by the linked flux)

$$= \frac{\mu i x^3}{2\pi r^4} dx \quad \text{via substitution}$$

$$\lambda_{int} = \int_0^r \frac{\mu i x^3}{2\pi r^4} dx$$

$$= \frac{\mu i}{8\pi}$$



Key Equations

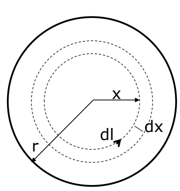
1. $\oint_c H \times dl = i$ 3. $\phi = \int_A B \cdot da$
2. $B = \mu H$

Internal Inductance

$$\lambda_{int} = \frac{\mu i}{8\pi}$$

$$\lambda_{int} = \frac{i}{2} x10^{-7} \text{ Wbt/m} \quad \text{evaluating constants}$$

$$L_{int} = \frac{1}{2} x10^{-7} \text{ H/m}$$



Key Equations

1. $\oint_c H \times dl = i$ 3. $\phi = \int_A B \cdot da$
2. $B = \mu H$

External Inductance

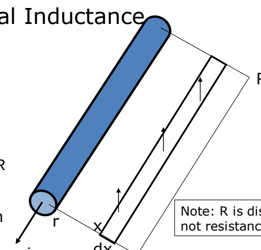
$$H2\pi x = i \quad \text{from eqn. 1}$$

$$B = \frac{\mu_0 i}{2\pi x} \quad \text{from eqn. 2}$$

$$\lambda_{ext} = \int_r^R B_x \ x \ dx \quad \text{between r an R}$$

$$= \mu_0 \int_r^R \frac{i}{2\pi x} dx \quad \text{via substitution}$$

Evaluate the integral



Note: R is distance, not resistance

Key Equations

1. $\oint_c H \times dl = i$ 3. $\phi = \int_A B \cdot da$
2. $B = \mu H$

External Inductance

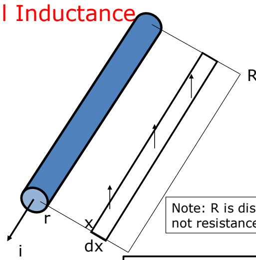
$H 2\pi x = i$ from eqn. 1

$B = \frac{\mu_0 i}{2\pi x}$ from eqn. 2

$\lambda_{ext} = \int_r^R B_x \times dx$ between r an R

$= \mu_0 \int_r^R \frac{i}{2\pi x} dx$ via substitution

$= \frac{\mu_0 i}{2\pi} \ln \frac{R}{r}$



Note: R is distance, not resistance

Key Equations

1. $\oint_c H \times dl = i$ 3. $\phi = \int_A B \cdot da$
2. $B = \mu H$

Inductance of a Infinite Wire

- Superposition of internal and external flux, inductance

$\lambda = 2 \times 10^{-7}\ i \left(\frac{\mu_r}{4} + \ln \frac{R}{r} \right)$ Wb-turn/m

internal external

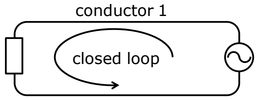
- Inductance is therefore:

$L = 2 \times 10^{-7} \left(\frac{\mu_r}{4} + \ln \frac{R}{r} \right)$ H/m

Note: derived flux linkages and inductance are per m

Single-Phase Inductance

- Next we consider single-phase circuit inductance
- Single phase: "hot" conductor and "return" conductor



Single-Phase Inductance

- Flux set up by conductor 1 at a distance greater than $D+r_2$ does not link the circuit
- Approximation: use D instead of $(D-r_1)$ or $(D-r_2)$
- L_1: inductance due to conductor 1 only

$L = 2 \times 10^{-7} \left(\frac{\mu_r}{4} + \ln \frac{R}{r} \right)$

$L_1 = \left(\frac{1}{2} + 2 \ln \frac{D}{r_1} \right) \times 10^{-7}$ H/m assuming $\mu_r = 1$

Single-Phase Inductance

$L_1 = \left(\frac{1}{2} + 2 \ln \frac{D}{r_1} \right) \times 10^{-7}$ H/m let's clean this up a bit

$L_1 = 2 \left(\ln e^{\frac{1}{4}} + \ln \frac{D}{r_1} \right) \times 10^{-7}$ H/m using $\ln e^{1/4} = \frac{1}{4}$

$L_1 = 2 \left(\ln \frac{D}{r_1 e^{\frac{-1}{4}}} \right) \times 10^{-7}$ H/m

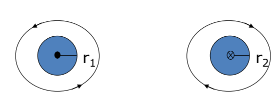
$L_1 = 2 \left(\ln \frac{D}{r_1'} \right) \times 10^{-7}$ H/m via substitution

$r_1' = r_1 e^{-\frac{1}{4}} = 0.778 r_1$

Interpretation: r_1' is the radius of a fictitious conductor assumed to have no internal flux, but with the same inductance of the actual conductor of radius r_1

Single-Phase Inductance

- What about the other conductor?
- Which direction is the flux going?
- Same direction: inductances can be added
- L is the loop inductance per m
- ½ L is the inductance per conductor



$L_2 = 2 \left(\ln \frac{D}{r_2'} \right) \times 10^{-7}$ H/m

$r' = r_1' = r_2'$ (standard assumption)

$L = L_1 + L_2 = 4 \left(\ln \frac{D}{r'} \right) \times 10^{-7}$ H/m

Example

Compute the inductance for a single-phase line that is 1km long, with a conductor radius of 0.02 m. The conductors are 3 m apart.

Example

Compute the inductance for a single-phase line that is 1km long, with a conductor radius of 0.02 m. The conductors are 3 m apart.

$r' = 0.02e^{-0.25} = 0.0156$ m

$L = 1000 \times 4\left(\ln\frac{D}{r'}\right)$ x 10^{-7} $=1000 \times 4\left(\ln\frac{3}{0.0156}\right)$ x 10^{-7}H=0.0021 H

What is the inductive reactance?

Example

Compute the inductance for a single-phase line that is 1km long, with a conductor radius of 0.02 m. The conductors are 3 m apart.

$r' = 0.02e^{-0.25} = 0.0156$ m

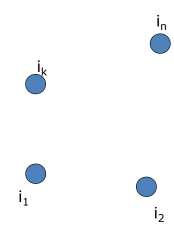
$L = 1000 \times 4\left(\ln\frac{D}{r'}\right)$ x 10^{-7} $=1000 \times 4\left(\ln\frac{3}{0.0156}\right)$ x 10^{-7}H=0.0021 H

What is the inductive reactance?

$X_L = 2\pi \times 60 \times 0.0021 = 0.79\Omega$

Many-Conductor Case

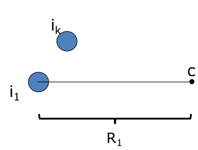
- Extend to multiple conductors: 1, …, k, … , n
- Assume distances between conductors is much greater than their radii
- Look at flux linkages of conductor 1 up to a radius R_1
- Add contributions of all other conductors



Note: n is a number, not the neutral

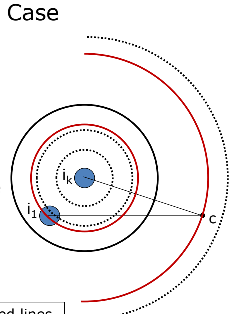
General Case

- Start by examining the flux linking conductor 1 up to radius R_1
- First consider the k^{th} conductor



General Case

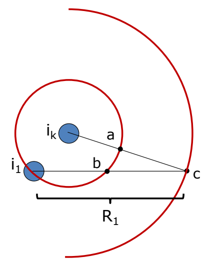
- Flux from current in k^{th} conductor form concentric circles around it
- Only some flux from i^k has a net contribution to the total flux between conductor 1 and c (distance R_1)
  - Solid lines: contribute to net flux
  - Dashed lines: no net flux contribution



Flux between red lines have net contribution

General Case

- Use point a to simplify the calculation
  - As R_1 increases, error decreases
- Therefore we consider the flux from conductor k between a and c

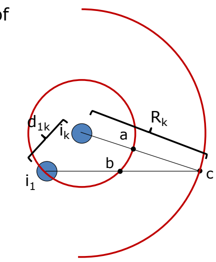


General Case

- Need distance from center of conductor k to point a, and to point c

$$\lambda_{1k} = 2\ln\left(\frac{R_k}{d_{1k}}\right) i_k \text{ x } 10^{-7}$$

- Next include other conductors



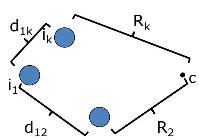
General Case

- Including all conductors

$$\lambda_1 = 2\left\{ \underbrace{i_1\left(\frac{1}{4} + \ln\frac{R_1}{r_1}\right)} + i_2 \ln\frac{R_2}{d_{12}} + \cdots + i_n \ln\frac{R_n}{d_{1n}} \right\} \text{ x } 10^{-7}$$

conductor 1 (internal + external)

  - R_x: distance from conductor x to point c
  - d_{1x}: distance from conductor 1 to conductor x



General Case

- What happens as $R_1 \rightarrow$ infinity?
- Assume:

  $i_1 + i_2 + \ldots + i_k \ldots + i_n = 0$ (is this reasonable?)
- Expanding:

$$\lambda_1 = 2\left\{ i_1\left(\frac{1}{4} + \ln\frac{R_1}{r_1}\right) + i_2 \ln\frac{R_2}{d_{12}} + \cdots + i_n \ln\frac{R_n}{d_{1n}} \right\} \text{ x } 10^{-7}$$

$$\lambda_1 = 2\left\{ i_1\left(\frac{1}{4} + \ln\frac{1}{r_1}\right) + i_2 \ln\frac{1}{d_{12}} + \cdots + i_n \ln\frac{1}{d_{1n}} \right\} \text{ x } 10^{-7} + 2\ i_1 \ln R_1 + i_2 \ln R_2 + \cdots + i_n \ln R_n \ \text{ x } 10^{-7}$$

using $\ln\frac{x}{y} = \ln\frac{1}{y} + \ln x$

General Case

- From previous slide:

$$\lambda_1 = 2\left\{ i_1\left(\frac{1}{4} + \ln\frac{1}{r_1}\right) + i_2 \ln\frac{1}{d_{12}} + \cdots + i_n \ln\frac{1}{d_{1n}} \right\} \text{ x } 10^{-7} + 2\ i_1 \ln R_1 + i_2 \ln R_2 + \cdots + i_n \ln R_n \ \text{ x } 10^{-7}$$

- Note that:

$$0 = i_1 + i_2 + \ldots + i_n$$
$$= -2\ln(R_1)(i_1 + i_2 + \ldots + i_n) \text{x } 10^{-7}$$
$$= -2\ i_1 \ln R_1 + i_2 \ln R_1 + \cdots + i_n \ln R_1 \ \text{ x } 10^{-7}$$ add this to λ_1

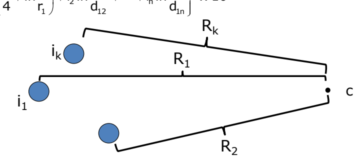
- Yields

$$\lambda_1 = 2\left\{ i_1\left(\frac{1}{4} + \ln\frac{1}{r_1}\right) + i_2 \ln\frac{1}{d_{12}} + \cdots + i_n \ln\frac{1}{d_{1n}} \right\} \text{ x } 10^{-7} + 2\left\{ i_1 \ln\frac{R_1}{R_1} + i_2 \ln\frac{R_2}{R_1} + \cdots + i_n \ln\frac{R_n}{R_1} \right\} \text{ x } 10^{-7}$$

General Case

- As $R_1 \rightarrow$ infinity, R_1 and R_x increase and become nearly equal and $\ln\left(\frac{R_k}{R_n}\right)$ becomes $\ln 1 = 0$
- Resulting in:

$$\lambda_1 = 2\left\{ i_1\left(\frac{1}{4} + \ln\frac{1}{r_1}\right) + i_2 \ln\frac{1}{d_{12}} + \cdots + i_n \ln\frac{1}{d_{1n}} \right\} \text{ x } 10^{-7}$$



General Case

- From last slide

$$\lambda_1 = 2\left\{ i_1\left(\frac{1}{4} + \ln\frac{1}{r_1}\right) + i_2 \ln\frac{1}{d_{12}} + \cdots + i_n \ln\frac{1}{d_{1n}} \right\} \text{ x } 10^{-7}$$

- via substitution

$$\lambda_1 = 2\left\{ i_1 \ln\frac{1}{r'_1} + i_2 \ln\frac{1}{d_{12}} + \cdots + i_n \ln\frac{1}{d_{1n}} \right\} \text{ x } 10^{-7}$$

- General kth conductor

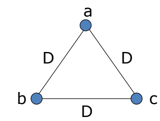
$$\lambda_k = 2\left\{ i_1 \ln\frac{1}{d_{k1}} + i_2 \ln\frac{1}{d_{k2}} + \cdots + i_k \ln\frac{1}{r'_k} + \cdots + i_n \ln\frac{1}{d_{kn}} \right\} \text{ x } 10^{-7}$$

Equilateral Spacing

Assume

$i_a + i_b + i_c = 0$

find inductance per meter



use $\lambda_1 = 2\left\{ i_1 \ln\frac{1}{r'_1} + i_2 \ln\frac{1}{d_{12}} + \cdots + i_n \ln\frac{1}{d_{1n}} \right\} \text{ x } 10^{-7}$

$$\lambda_a = 2\left\{ i_a \ln\frac{1}{r'_1} + i_b \ln\frac{1}{D} + i_c \ln\frac{1}{D} \right\} \text{ x } 10^{-7}$$

$$\lambda_a = 2\left\{ i_a \ln\frac{D}{r'_1} \right\} \text{ x } 10^{-7} \text{ Wbt/m}$$

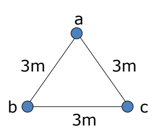
using: $i_b \ln\frac{1}{D} + i_c \ln\frac{1}{D} = i_a \ln D$

$$L_a = 2\left\{ \ln\frac{D}{r'_1} \right\} \text{ x } 10^{-7} \text{ H/m}$$

inductance per phase

Example

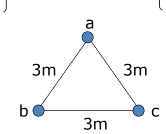
Compute the a-phase inductance of the line shown below if each conductor has a radius of 0.02m. Assume the line is 1km in length.



Example

Compute the a-phase inductance of the line shown below if each conductor has a radius of 0.02m. Assume the line is 1km in length.

$$L_a = 1000 \times 2\left\{ \ln\frac{D}{r'_1} \right\} \text{ x } 10^{-7} = 1000 \times 2\left\{ \ln\frac{3}{0.156} \right\} \text{ x } 10^{-7} = 0.00105\text{H}$$



Observations

- For equilateral spacing and balanced conditions, only self inductances need to be modeled (no mutual inductance)
- Decreasing inductance can be accomplished by decreasing separation between phases or increasing the conductor radi

Line Inductance Summary

- Developed expressions for internal and external inductance
- For single-phase lines

$$L = 2 \text{ x } 10^{-7} \left(\frac{\mu_r}{4} + \ln\frac{R}{r} \right)$$

- For equilateral (symmetrical) spacing

$$L = 4\left(\ln\frac{D}{r'} \right) \text{ x } 10^{-7} \text{ H/m}$$

$$L_a = 2\left\{ \ln\frac{D}{r'_1} \right\} \text{ x } 10^{-7} \text{ H/m}$$