07-Nodal Analysis

Text: 3.1 – 3.4

ECEGR 210

Electric Circuits I

Overview

- Introduction
- Nodal Analysis
- Nodal Analysis with Voltage Sources

Introduction

- Basic Circuit Laws
  - Ohm’s Law
  - Kirchhoff’s Voltage Law (KVL)
  - Kirchhoff’s Current Law (KCL)
- Now we seek to analyze any linear circuit systematically using
  - Nodal analysis
  - Mesh analysis

Introduction

- Express circuit as a system of linear equations
- Solve linear equations
  - Substitution
  - Gaussian elimination
  - Cramer's Rule
  - Numerical methods
  - Others

$$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Remember: for a unique solution there must be an equal number of independent equations as unknowns

Introduction

- Let X be a vector of variables to be solved for
- Nodal Analysis:
  - X: node voltages
  - Y: conductances
  - b: currents
- Mesh Analysis:
  - X: loop currents
  - Y: resistances
  - b: voltages

$$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

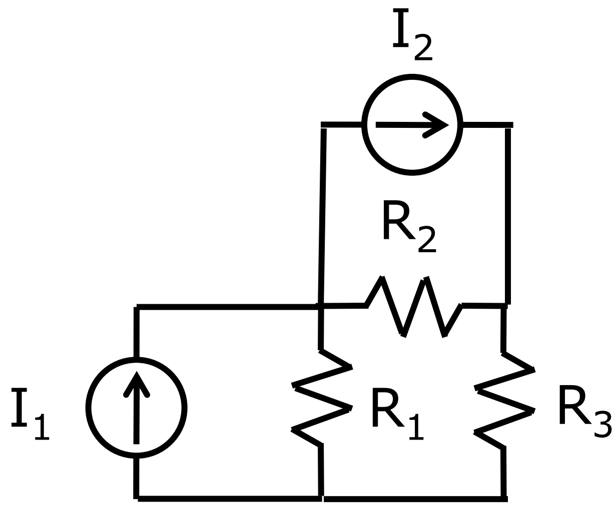
Nodal Analysis

Steps to solve a circuit with N nodes

1. Assign a reference node
2. Assign a variable to the voltages at all nodes wrt the reference node (N-1 variables)
3. Apply KCL to each of the N-1 non-reference nodes to generate N-1 equations
4. Use Ohm's Law to express currents as functions of node voltages
5. Solve resulting simultaneous equations

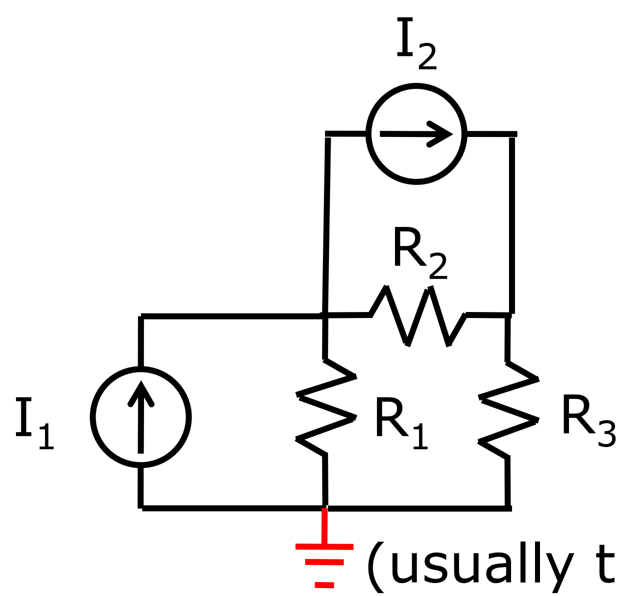
Nodal Analysis

- Let I_1, I_2, R_1, R_2, R_3 be known (given)
- Solve for the voltages at each node



Step 1: Assign Reference Node

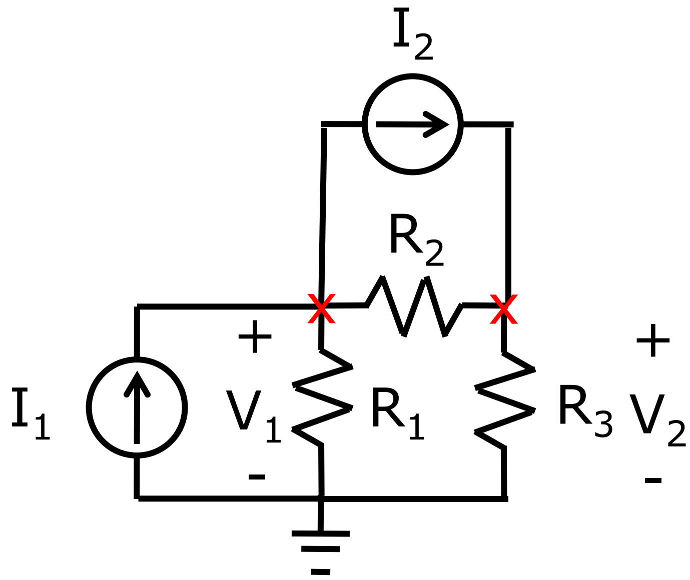
- Recall: all voltage is a relative measure
- Reference node is also known as *ground*
  - In power systems it is actually the ground (earth)
  - In electronics it could be the metallic chassis
  - Or arbitrary



(usually the “bottom” of circuit)

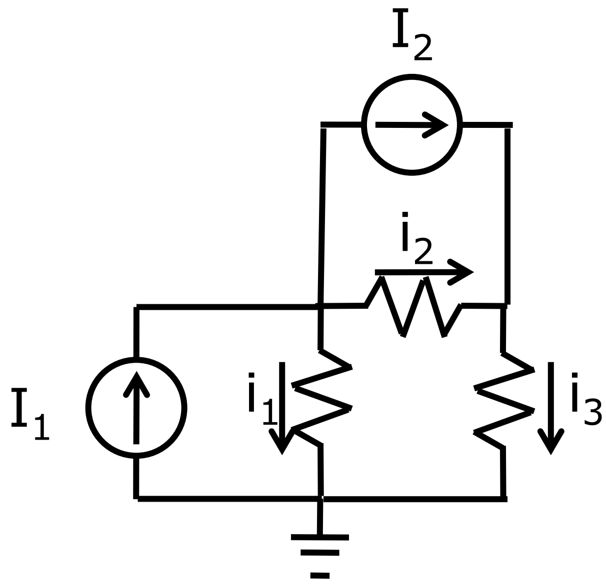
Step 2: Assign Variables to Nodes

- There are two remaining nodes (variables)
- Assign voltages and polarities wrt reference node
- No need to write voltage across R_2
  - Does not add an independent equation (no new information)
  - Can solve later as $V_1 - V_2$



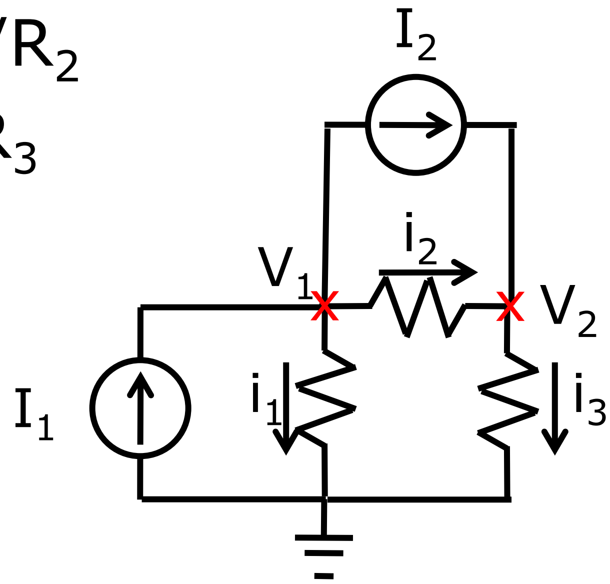
Step 3: Apply KCL

- KCL gives one equation per node (two equations)
- $I_1 = I_2 + i_1 + i_2$
- $I_2 = i_3 - i_2$



Step 4: Apply Ohm's Law

- Need to write i_1, i_2, i_3 in terms of the node voltages
- Remember sign convention
- $i_1 = (V_1 - 0)/R_1$
- $i_2 = (V_1 - V_2)/R_2$
- $i_3 = (V_2 - 0)/R_3$



Step 5: Solve Equations

- Recap
  - two variables (V_1, V_2)
  - two KCL equations
- KCL equations use intermediate variables i_1, i_2, i_3
  - $I_1 = I_2 + i_1 + i_2$
  - $I_2 = i_3 - i_2$
- Use substitution to express KCL in terms of voltages
  - $i_1 = (V_1 - 0)/R_1$
  - $i_2 = (V_1 - V_2)/R_2$
  - $i_3 = (V_2 - 0)/R_3$

Step 5: Solve Equations

- Via substitution:
  - $I_1 = I_2 + V_1/R_1 + (V_1 - V_2)/R_2$
  - $I_2 = V_2/R_3 - (V_1 - V_2)/R_2$
- Can we solve this system of equations?
- Yes!
  - Two independent equations, two unknowns
  - The rest is just math

Step 5: Solve Equations

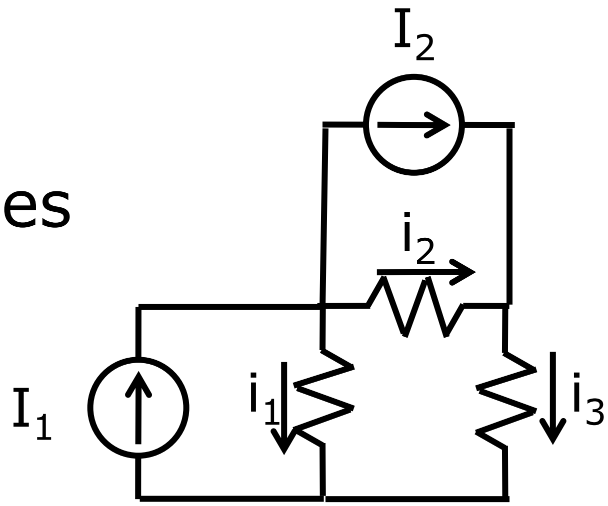
- In matrix form
  - $I_1 = I_2 + V_1/R_1 + (V_1 - V_2)/R_2$
  - $I_2 = V_2/R_3 - (V_1 - V_2)/R_2$

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_3} + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

- Many methods of solving linear equations

Example

- Let
  - $I_1 = 10A$
  - $I_2 = 5A$
  - $R_1 = 6\Omega$
  - $R_2 = 4\Omega$
  - $R_3 = 2\Omega$
- Find all voltages



Example

- Equations:
  - $I_1 = I_2 + V_1/R_1 + (V_1 - V_2)/R_2$
  - $I_2 = V_2/R_3 - (V_1 - V_2)/R_2$

- Using circuit values:
  - $10 = 5 + V_1/6 + (V_1 - V_2)/4$
  - $5 = V_2/2 - (V_1 - V_2)/4$

Example

- Starting with the second equation
  - $5x4 = [V_2/2 - (V_1 - V_2)/4]x4$

  Yields

  $20 = 2V_2 - (V_1 - V_2) = 3V_2 - V_1$

  Now the first equation:

  $10x12 = [5 + V_1/6 + (V_1 - V_2)/4]x12$

  $120=60 + 2V_1 + 3V_1 - 3V_2$

  $60 = 5V_1 - 3V_2$

Example

- $20 = -V_1 + 3V_2$
- $60 = 5V_1 - 3V_2$
- Solve using elimination
  - $20x5 = [-V_1 + 3V_2]x5 = 100 = -5V_1 + 15V_2$

$$60 = 5V_1 - 3V_2$$

$$160 = 0 + 12V_2$$

- Yields:
  - $V_2 = 13.333V$
  - $V_1 = 20V$

Example

- Solving in matrix form:

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_3} + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{6} + \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} + \frac{1}{4} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Can use Matlab, Gaussian elimination
Cramer's Rule, matrix inversion

Solution by Matrix Inversion

$$\begin{bmatrix} \frac{1}{6}+\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2}+\frac{1}{4} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

G (matrix)

$$\mathbf{G} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

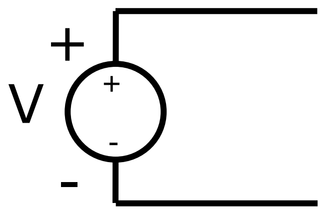
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \mathbf{G}^{-1} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 13.3 \end{bmatrix}$$

where $\mathbf{G}^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 1.67 \end{bmatrix}$

For small matrices use "inv()" command in Matlab

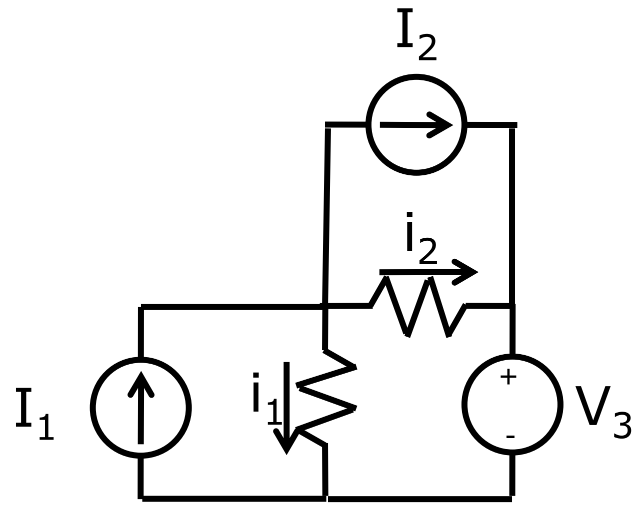
Nodal Analysis with Voltage Sources

- Nodal analysis requires knowledge of current through elements
- Recall that the current through a voltage source is not readily known
- Example:
  - Current through resistor is $I = (V_a - V_b)/R$
  - But what is the current through a voltage source? How is the "R" in the equation to be interpreted?



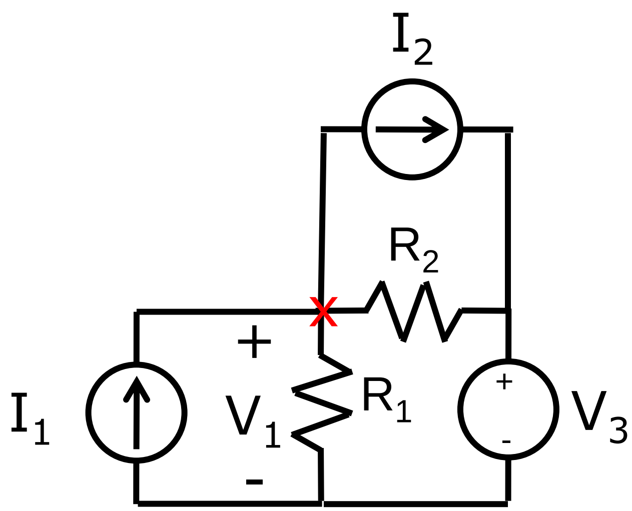
Nodal Analysis with Voltage Sources

- If the voltage source is connected to the reference node, then it can be easily included in nodal analysis



Nodal Analysis with Voltage Sources

- Only one node with unknown voltage
  - $I_1 = I_2 + V_1/R_1 + (V_1 - V_3)/R_2$
- Compare this to earlier example (2 eqns, 2 unknowns)



Nodal Analysis with Voltage Sources

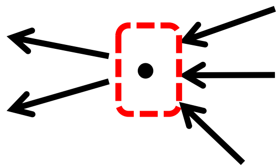
- If voltage source not connected to reference:
  - Move the reference so it is! (only works if there is one voltage source)

  Or

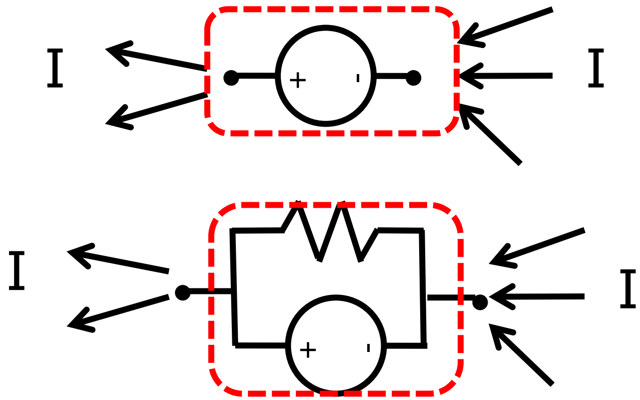
  - Use Supernode concept

Nodal Analysis with Voltage Sources

- Supernode: a closed surface containing a voltage source and its two nodes AND <u>any elements in parallel with it</u>



node



Supernodes

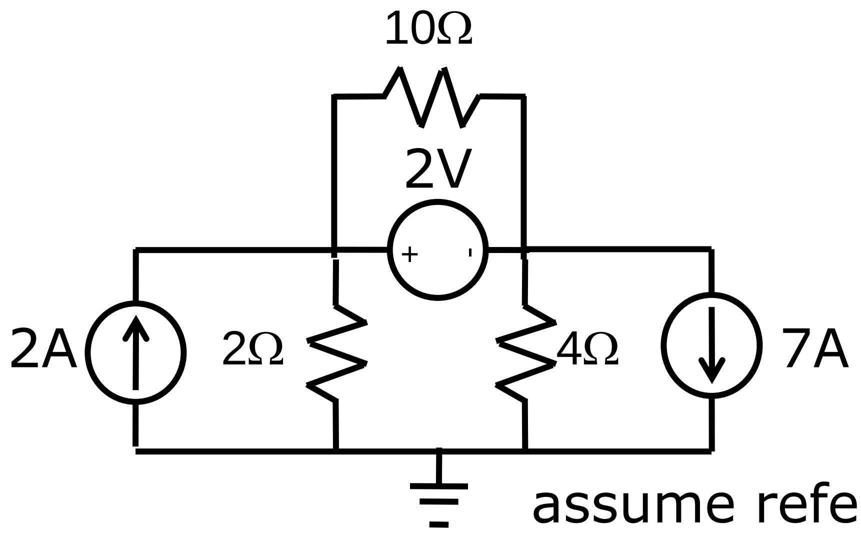
- Current into supernode = current out of supernode
- Also applies to dependent voltage sources

Nodal Analysis with Voltage Sources

- Properties of a supernode
  - Voltage source inside the supernode provides a constraint equation needed to solve for node voltages
  - A supernode has no voltage of its own
  - There is only one KCL equation for the supernode, but two unknown node voltages
  - Additional equation is found from application of KVL

Nodal Analysis with Voltage Sources

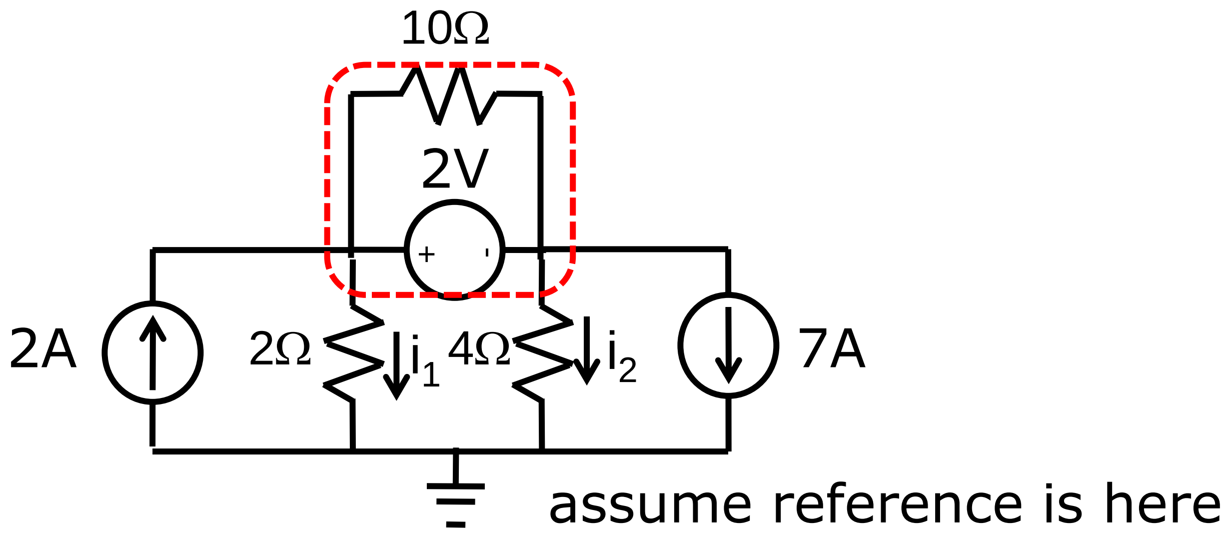
- Find the node voltages



assume reference is here

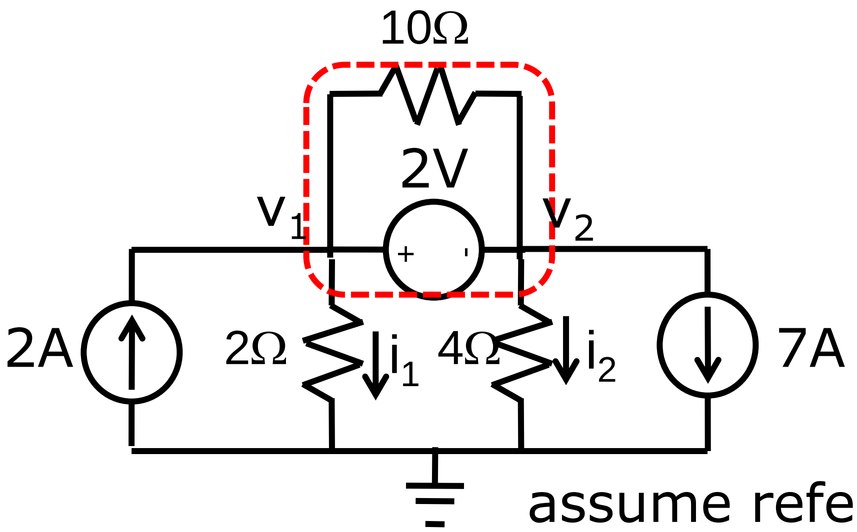
Nodal Analysis with Voltage Sources

- Make a supernode out of the voltage source
- Write KCL for supernode
  - $2 = i_1 + i_2 + 7$ (only one equation since it is a supernode)



Nodal Analysis with Voltage Sources

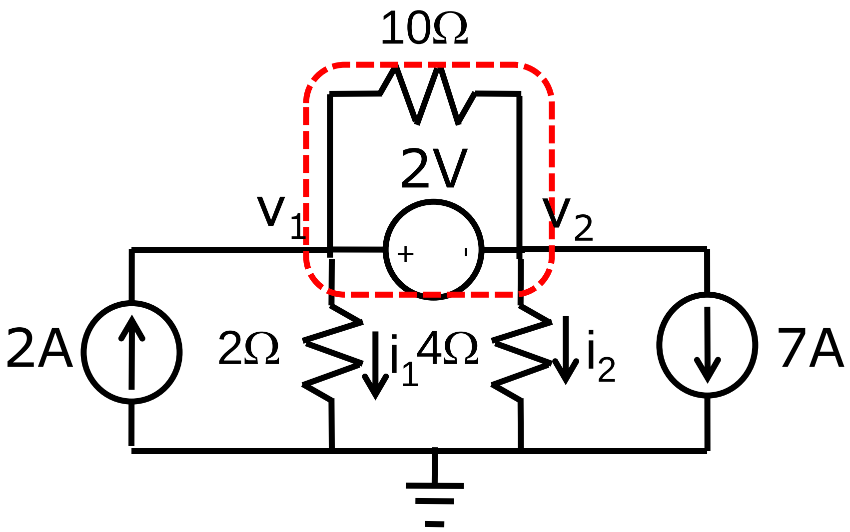
- Write i_1, i_2 in terms of node voltage
- $i_1 = (v_1 - 0)/2$
- $i_2 = (v_2 - 0)/4$



assume reference is here

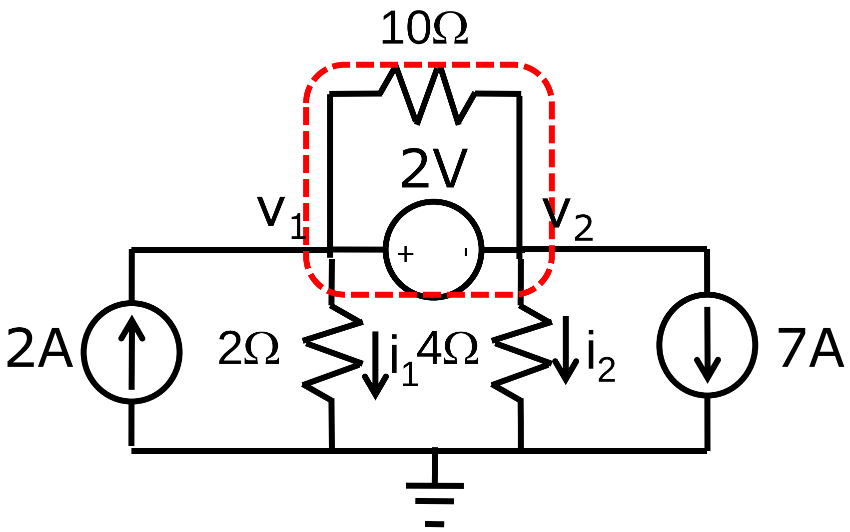
Nodal Analysis with Voltage Sources

- Substitute into supernode KCL
- $2 = i_1 + i_2 + 7$ (supernode KCL)
  - $2 = 0.5v_1 + 0.25v_2 + 7$ (after substitution)
  - $8 = 2v_1 + v_2 + 28$ (multiplying by 4)



Nodal Analysis with Voltage Sources

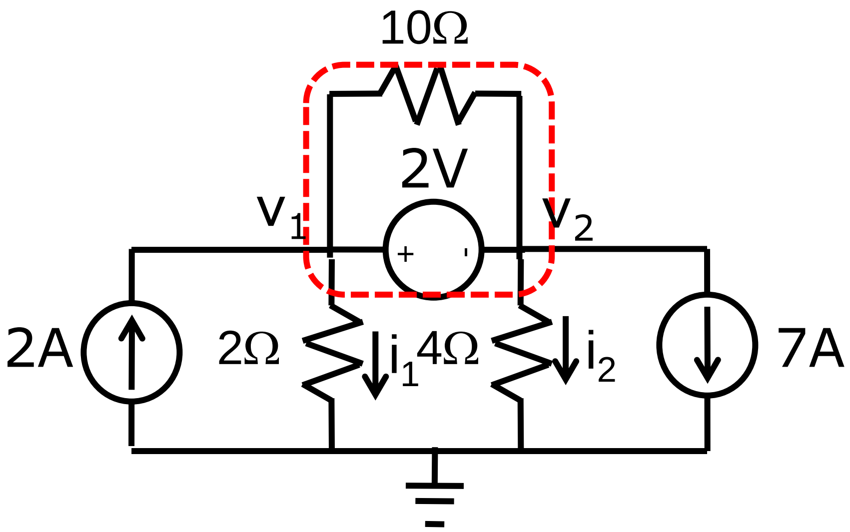
- Apply KVL around loop
  - $-v_1 + 2 + v_2 = 0$ (second equation)



Nodal Analysis with Voltage Sources

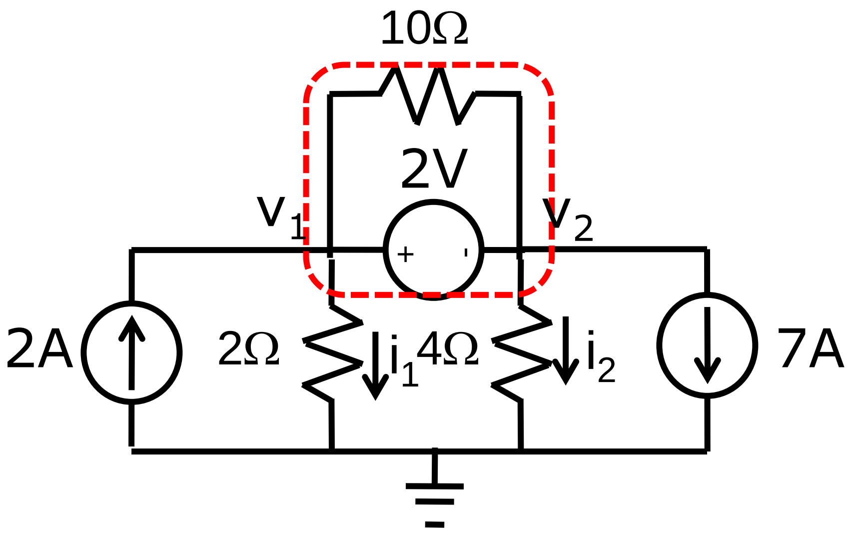
- How many independent equations?
  - $8 = 2v_1 + v_2 + 28$
  - $-v_1 + 2 + v_2 = 0$
- How many unknowns?
  - v_1, v_2

Solve!



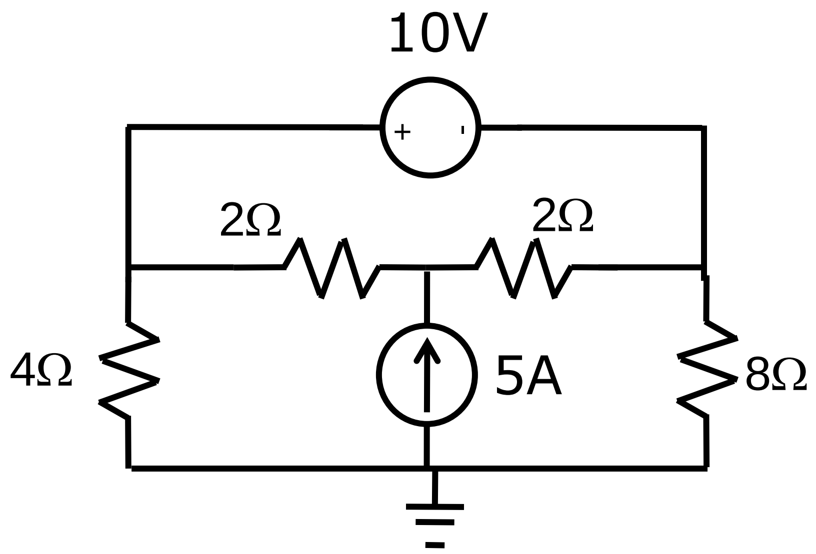
Nodal Analysis with Voltage Sources

- Solving…
  - $8 = 2v_1 + v_1 - 2 + 28$
  - $v_1 = -5.333V$
  - $v_2 = -7.333V$



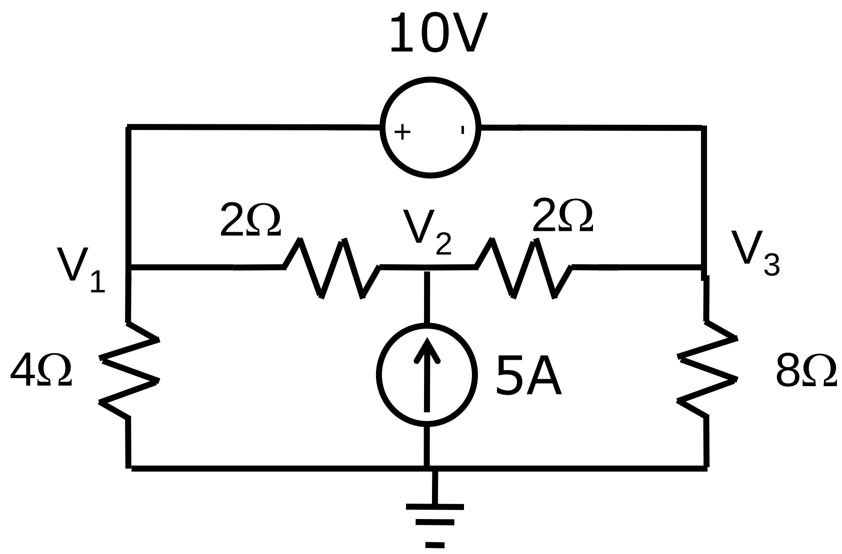
Example

- Find the node voltages in the circuit shown



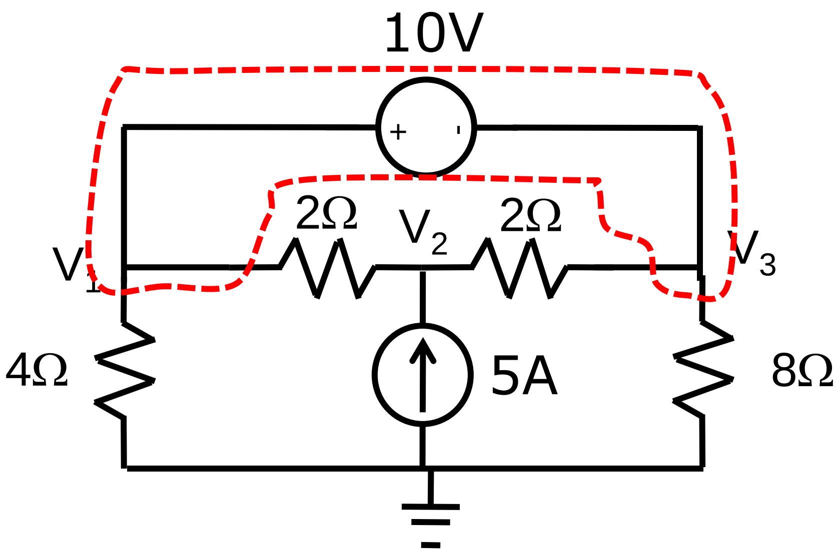
Example

- 3 unknown voltages: V_1, V_2, V_3
- Need three independent equations (that do not introduce new variables)



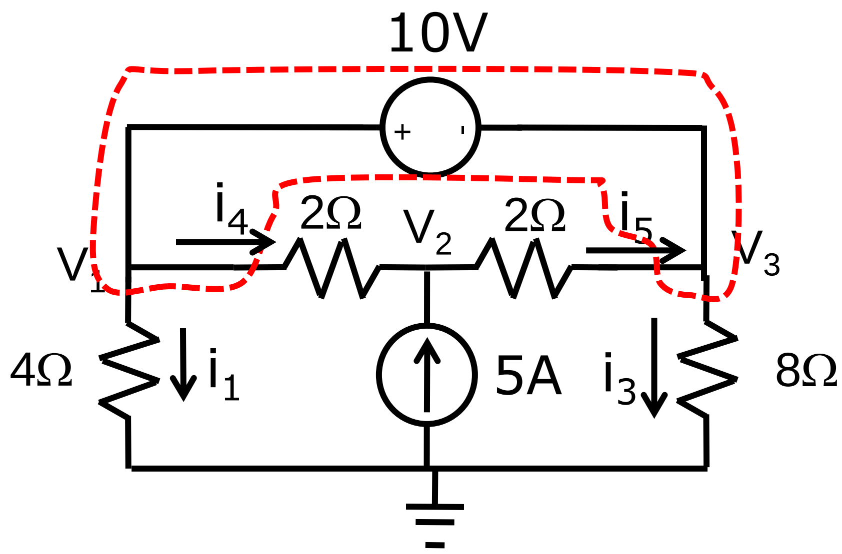
Example

- Identify the supernode



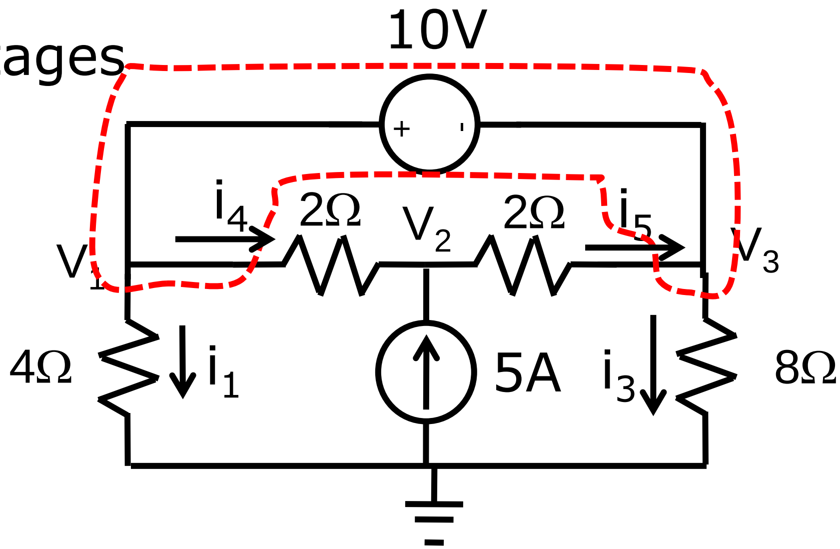
Example

- Write KCL for the supernode
  - $i_5 = i_1 + i_4 + i_3$ (supernode)
- Write KCL for the other node
  - $5 + i_4 = i_5$ (node 2)



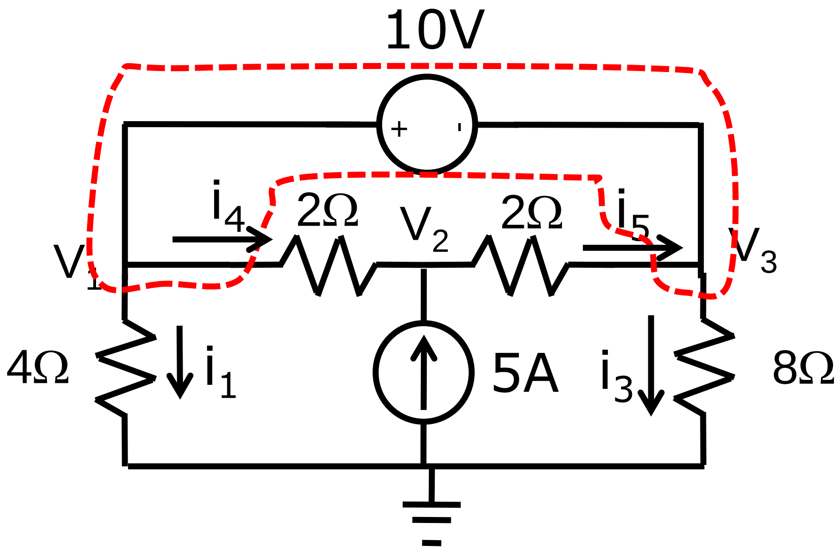
Example

- Express currents in terms of voltages
  - $i_1 = (V_1 - 0)/4$
  - $i_3 = (V_3 - 0)/8$
  - $i_4 = (V_1 - V_2)/2$
  - $i_5 = (V_2 - V_3)/2$



Example

- Have two equations
  - KCL for node 2, supernode
- Need one more independent equation
- Do KVL around top loop
  - $10 = 2i_4 + 2i_5$



Example

- $5 + i_4 = i_5$ (node 2)
- $i_6 = i_1 + i_5 + i_3$ (supernode)
  - $i_1 = (V_1 - 0)/4 = 0.25V_1$
  - $i_3 = (V_3 - 0)/8 = 0.125V_3$
  - $i_4 = (V_1 - V_2)/2 = 0.5V_1 - 0.5V_2$
  - $i_5 = (V_2 - V_3)/2 = 0.5V_2 - 0.5V_3$

$5 + 0.5V_1 - 0.5V_2 = 0.5V_2 - 0.5V_3$ (solving node 2 eqn)

$10 + V_1 - V_2 = V_2 - V_3$

$0.5V_2 - 0.5V_3 = 0.25V_1 + 0.5V_1 - 0.5V_2 + 0.125V_3$ (supernode eqn)

$4V_2 - 4V_3 = 2V_4 + 4V_1 - 4V_2 + V_3$

$10 = 2i_4 + 2i_5$ (KVL of top loop)

$10 = V_1 - V_2 + V_2 - V_3$

Example

$10 + V_1 - V_2 = V_2 + V_3$ (node 2)

$10 = -V_1 + 2V_2 + V_3$ (after rearranging)

$4V_2 - 4V_3 = 2V_1 + 4V_1 - 4V_2 - V_3$ (supernode)

$0 = 6V_1 - 8V_2 + 3V_3$ (after rearranging)

$10 = V_1 - V_2 + V_2 - V_3$ (KVL of top loop)

$10 = V_1 + 0V_2 - V_3$ (after rearranging)

Solving

$V_1 = 12.22V$

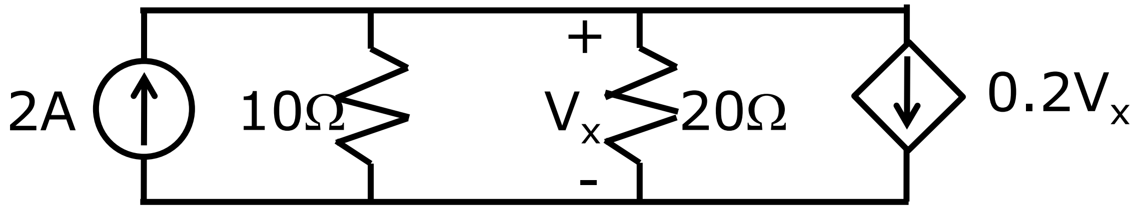
$V_2 = 10V$

$V_3 = 2.22V$

$$\begin{bmatrix} -1 & 2 & 1 \\ 6 & -8 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 10 \end{bmatrix}$$

Example

- Find V_x



Example

- One node, write equation from KCL
  - $2 = I_1 + I_2 + 0.2V_x$
- From Ohm's Law
  - $I_1 = V_x/10$
  - $I_2 = V_x/20$
- Solving:
  - $2 = 0.1V_x + 0.05V_x + 0.2V_x$
  - $V_x = 5.71V$